Journal of Scientific Research & Reports
6(2): 165-171, 2015; Article no.JSRR.2015.141
ISSN: 2320-0227

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Fully Implicit Hybrid Block –Predictor Corrector Method for the Numerical Integration of $y''' = f(x, y, y', y''), y(x_0) = \eta_0, y'(x_0) = \eta_1, y''(x_0) = \eta_3$

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/JSRR/2015/14822
Editor(s):
(1) Narayan Thapa, Department of Mathematics and Computer Science, Minot State University, 58707 Minot, ND, USA.
Reviewers:
(1) Jianfei Huang, College of Mathematics, Qingdao University, China.
(2) Anonymous, Nigeria.
Complete Peer review History: http://www.sciencedomain.org/review-history.php?iid=963&id=22&aid=8209

Original Research Article

Received 23rd October 2014
Accepted 29th December 2014
Published 21st February 2015

ABSTRACT

A fully implicit hybrid Block- Predictor Corrector method for the numerical integration of initial value problems of third order ordinary differential equations is presented in this paper. We adopted the approach of collocation approximation in the derivation of the scheme to generate a scheme with continuous coefficients, from where additional schemes were developed. The implementation strategy involves combination of the main scheme and other additional schemes as simultaneous Integrator to initial value problems of third order ordinary differential equations. Properties analysis of the block method showed that it is consistent, convergent, zero stable and absolutely stable. Numerical examples were given.

Keywords: Block predictor corrector; fully implicit; hybrid; numerical integration; third order ordinary differential equations.

1. INTRODUCTION

The initial value problems of third order ordinary differential equation:

$$y''' = f(x, y, y', y''), y(x_0) = \eta_0, \quad y'(x_0) = \eta_1, y''(x_0) = \eta_3 \tag{1}$$

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is considered. It is assumed that the numerical solution to (1) is required on a given set of mesh:

$$\Pi = \{x_n / x_n = a + nh, h = x_{n+1} - x_n, n = 0,1,2,..N\}$$

Where $N = \dfrac{b-a}{h}$.

Many researchers have worked on the numerical solution of (1), they include: [1-8,9] to mention but just a few. The works of [9,10] have their implementation strategy being predictor - corrector mode. The drawbacks of the Predictor – Corrector methods are well known, as discussed in [11]; they are not self-starting; they advance the numerical integration of the ordinary differential equations in one-step at a time, which leads to overlapping of the piecewise polynomials solution model; [12]. It was observed that, the overlapping creates a disadvantage because the numerical model fails to represent the solution uniquely elsewhere than the mesh-points. [13] observed that for the numerical solution of boundary value problems, this is the major criticism of the linear multistep methods in favor of the finite element methods.

The works of [8,5,2,3,4] whose implementation strategy is block mode addressed the afore – mentioned draw backs of the Linear multi step method whose implementation is Predictor – Corrector mode.

It is pertinent to note that [2] advanced the course of study in the proposition of block method by developing a single step implicit hybrid block method for the numerical solution of initial value problems of third order ordinary differential equations. [3,4] went further in the study of numerical solution of initial value problems of third order ordinary differential equations using single step method by proposition of two schemes that were found to be efficient, adequate and suitable towards catering for the class of problem- higher order ordinary differential equations - for which they were designed. The contribution to knowledge of the afore – mentioned works is that a single step method was shown and effectively tested to be capable of solving higher order ordinary differential equations, against the initial believe, before now, that single step numerical methods were only capable of solving first order ordinary differential equations.

Consequently, in this work, we are motivated to go further in the proposition of a single step method for the direct numerical solution of higher order ordinary differential equations; thus, we increase the partitioning of the single step lenght (x_n, x_{n+1}) into five sub steps, having initially considered cases three and four substeps in the initial works of [3,4] to give rise to a fully Implicit block linear multi step scheme for the solution of initial value problems of third order ordinary differential equations. The merit of this is the elimination of the use of predictors by the provision of sufficiently accurate simultaneous difference equations from a single continuous formula and its derivatives.

The general block formula is given by:

$$Y_m = ey_n + h^{\mu} df(y_n) + h^{\mu} bF(y_m) \tag{2}$$

where e is $s \times s$ vector, d is r - vector and b is $r \times r$ vector, s is the interpolation points and r is the *collection points. F is a k – vector whose* J^{th} entry is $f_{n+j} = f(t_{n+j,} y_{n+j})$, μ is the order of the differential equation [14].

Given a predictor equation in the form:

$$Y_m^{(0)} = ey_n + h^{\mu} df(y_n). \tag{3}$$

By Putting (3) in (2) we have:

$$Y_m = ey_n + h^{\mu} df(y_n) + h^{\mu} bF(ey_n + h^{\mu} dfy_n). \tag{4}$$

According to [16,17], equation (4) is called a self starting block-predictor-corrector method because the prediction equation is gotten directly from the block formula.

Consequently, our focus in this paper is the proposition of a fully implicit continuous hybrid block – Predictor corrector algorithm with a single step length for the numerical solution of third order ordinary differential equations.

2. METHODOLOGY

2.1 Derivation of the Continuous Coefficients

We take our basis function to be a power series of the form:

$$y(x) = \sum_{j=0}^{r+s} a_j x^j \tag{5}$$

We obtain the third derivative of (5) as:

$$y'''(x)=\sum_{j=0}^{r+s} j(j-1)(j-2)a_j x^{j-3} \tag{6}$$

By putting (6) into (1) we have the differential system:

$$\sum_{j=0}^{r+s} j(j-1)(j-2)a_j x^{j-3} = f(x, y(x), y'(x), y''(x)) \tag{7}$$

Where a_j are the parameters to be determined, while r+s denotes the number of collocation and interpolation points. By collocating (7) at the mesh points $x = x_{n+j}, j = 0(1/5)1$, and interpolating (5) at $x = x_{n+j}$, $j = 2/5 (3/5) 4/5$ yields a system of equations:

$$\sum_{j=0}^{r+s} a_j x^j = y_{n+s} \tag{8}$$

$$\sum_{j=0}^{r+s} j(j-1)(j-2)a_j x^{j-3} = f_{n+r} \tag{9}$$

By putting these system of equations in matrix form and then solved to obtain the values of parameters a_j 's , j = 0, $1/5$, .. which when substituted in (5), yields, after some manipulation, an hybrid linear method with continuous coefficients of the form:

$$y(x)=\sum_{j=0}^{1}\alpha_j y_{n+j}(x)+h^3\sum_{j=0}^{1}\beta_j f_{n+j}(x) \tag{10}$$

The co efficient of $\alpha_j(x)$ and β_j are:

$$\alpha_{1/5}(t)=1/8\left(t^2+4t+3\right)$$

$$\alpha_{2/5}(t)=-1/4\left(\left(t^2+2t+9\right)\right)$$

$$\alpha_{3/5}(t)=1/8\left(25t^2-1\right)$$

$$\alpha_{4/5}(t)=1/4\left(\left(t^2-4t-1\right)\right)$$

$$\beta_0(t)=\frac{1}{60{,}000}(7t^8+26t^7-64t^6+320t^5-340t^4+256t^3-312t+106)$$

$$\beta_{1/5}(t)=\frac{1}{60{,}000}(5t^8-3584t^7+124t^6+320t^5-2540t^4+2002t^3-2326t+6006)$$

$$\beta_{2/5}(t)=-\frac{1}{60{,}000}(17t^8+36t^7-72t^6+328t^5-341t^4+266t^3-322t+106)$$

$$\beta_{3/5}(t)=\frac{1}{60{,}000}(84t^8+216t^7-64t^6+338t^5-440t^4+1256t^3-1212t+84)$$

$$\beta_{4/5}(t)=-\frac{1}{60{,}000}(98t^8+326t^7-372t^6+428t^5-441t^4+266t^3-428t+130)$$

$$\beta_1(t)=\frac{1}{60{,}000}(104t^8+1316t^7-64t^6+345t^5-1440t^4+1256t^3-1312t+24)$$

Where $t=\dfrac{x-x_n}{h}$

2.2 Derivation of the Block Method

The general block formula proposed by [14], in the Normalized form is given by:

$$A^{(0)}Y_m = ey_n + h^{\mu-\lambda}df(y_n)+h^{\mu-\lambda}bF(y_m) \tag{11}$$

By evaluating (10) at $t=1, 1/5, 0$; the first, second and the third derivative at $x = x_{n+1}, i = 0(1/5)1$ and substituting into (10) gives the coefficients of (11) as:

$$d=\begin{bmatrix}\begin{bmatrix}\frac{350859600}{1786680000} & \frac{-352800}{127620000} & \frac{12297600}{595560000} & \frac{30038400}{357336000} & \frac{-17640000}{783820800} & \frac{2436276}{357336000} & \frac{36892800}{714672000} & \frac{-2462510}{357336000}\end{bmatrix} \\ \frac{-11067400}{714672000} \quad \frac{184464000}{714672000} \quad \frac{36892800}{714672000} \quad \frac{4611600000}{8933400000} \quad \frac{36892800}{71467200} \quad \frac{36892800}{71467200} \quad \frac{36892800}{71467200}\end{bmatrix}^T$$

$$e = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 2 & \frac{9}{2} & 8 & \frac{25}{2} & 1 & 2 & 3 & 4 & 5 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T$$

$A^0 = 15 \times 15$ identity matrix

$$B = \begin{bmatrix}
\frac{172885078}{1786680000} & \frac{3734185}{1276200000} & \frac{130755361}{595560000} & \frac{150415585}{357336000} & \frac{19119722}{127620000} \\
\frac{3175714736}{1786680000} & \frac{7668054}{127620000} & \frac{343724486}{595560000} & \frac{272823798}{357336000} & \frac{36277080}{12760000} \\
\frac{2076735693}{1786680000} & \frac{4971676}{127620000} & \frac{158856938}{595560000} & \frac{178805422}{357336000} & \frac{22790936}{127620000} \\
\frac{10610219}{1786680000} & \frac{22185}{127620000} & \frac{817589}{595560000} & \frac{962805}{357336000} & \frac{-442095}{127620000} \\
\frac{138314508}{1786680000} & \frac{248736}{1786680000} & \frac{9574303}{595560000} & \frac{11697815}{357336000} & \frac{1233040}{127620000} \\
\frac{6419633}{357336000} & \frac{-4186924}{357336000} & \frac{1327592}{357336000} & \frac{3816391}{357336000} & \frac{52713409}{357336000} \\
\frac{183536001}{714672000} & \frac{298759110}{714672000} & \frac{191245626}{714672000} & \frac{19712842}{714672000} & \frac{-42576915}{714672000} \\
\frac{25991823}{357336000} & \frac{53676378}{357336000} & \frac{32700256}{357336000} & \frac{162385}{357336000} & \frac{1593673}{357336000} \\
\frac{138619083}{714672000} & \frac{262365922}{714672000} & \frac{170033580}{714672000} & \frac{10944615}{714672000} & \frac{-19712842}{714672000} \\
\frac{2379700935}{714672000} & \frac{4514195874}{714672000} & \frac{2919421398}{714672000} & \frac{12859779}{714672000} & \frac{201350125}{714672000} \\
\frac{293340490}{714672000} & \frac{533513724}{714672000} & \frac{331659272}{714672000} & \frac{1782666}{714672000} & \frac{22222724}{714672000} \\
\frac{4611600000}{8933400000} & \frac{3603463405}{8933400000} & \frac{6542807040\ 6}{8933400000} & \frac{4112902978}{8933400000} & \frac{221354985}{8933400000} \\
\frac{28794746}{71467200} & \frac{51540870}{71467200} & \frac{337668756}{71467200} & \frac{1229646}{71467200} & \frac{21709408}{71467200} \\
\frac{288241362}{71467200} & \frac{514966284}{71467200} & \frac{327524384}{71467200} & \frac{1612506}{71467200} & \frac{16037408}{71467200} \\
\frac{286524164}{71467200} & \frac{519750616}{71467200} & \frac{339282440}{71467200} & \frac{6327356}{71467200} & \frac{41643652}{71467200} \\
\frac{80940746}{71467200} & \frac{94200746}{71467200} & \frac{207468794}{71467200} & \frac{8190744}{71467200} & \frac{2499419}{71467200}
\end{bmatrix}^T$$

3. ANALYSIS OF THE PROPERTIES OF THE BLOCK

3.1 Order of the Method

The linear operator of the block (11) is defined as:

$$L\{y(x):h\} = Y_m - ey_m + h^{\mu-\lambda} df(y_m) + h^{\mu-\lambda} bF(y_m) \quad (12)$$

By expanding $y(x_n + ih)$ and $f(x_n + jh)$ in Taylor series, (12) becomes:

$$L\{y(x):h\} = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \ldots + C_p h^p y^{(p)}(x) + \quad (13)$$

The block (11) and associated linear operator are said to have order p if

$$C_0 = C_1 = \ldots = C_{p+1} = 0, C_{p+2} \neq 0.$$

The term C_{p+2} is called the error constant and implies that the local truncation error is given by:

$$t_{n+k} = C_{p+2}h^{(p+2)}y^{(p+2)}(x_n) + 0h^{(p+3)} \quad (14)$$

Hence the block (11) has order 7 with error constant:

$$C_{p+2} = \begin{bmatrix} \frac{31421}{82900}, \frac{6023}{76800}, \frac{-1789}{16640}, \frac{3247}{28319}, \frac{2647}{27620} \\ \frac{6427}{71420}, \frac{4417}{13560}, \frac{1427}{78200}, \frac{6836}{12894}, \frac{3914}{78419} \\ \frac{9104}{81019}, \frac{2985}{15923}, \frac{1362}{16542}, \frac{3255}{15057}, \frac{2165}{81627} \end{bmatrix}$$

3.2 Zero Stability of the Block

The block (11) is said to be Zero stable if the roots $z_s = 1,2,...,N$ of the characteristic polynomial $\rho(z) = \det(zA - E)$, satisfies $|z| \le 1$ and the root $|z| = 1$ has multiplicity not exceeding the order of the differential equation. Moreover as

$$h^{\mu} \to 0, \rho(z) = z^{r-\mu}(\lambda - 1),$$

Where μ is the order of the differential equation, for the block (11), $r = 15, \mu = 3$

$$\rho(z) = \lambda^{12}(\lambda - 1)^3$$

Hence our method is Zero stable.

3.3 Convergence

According to [15], the necessary and sufficient condition for a numerical method to be convergent is for it to be Zero stable and has order $p \ge 1$, from the above condition, it could be seen that our method is convergent.

4. NUMERICAL EXPERIMENTS

To test the accuracy, workability and suitability of the method, we adopted our method to solving some initial value problems of third order ordinary differential equations.

4.1 Implementation Strategy

The block formula which is a system of equations expressed in compact (block) form Eq. 11 is used as simultaneous integrator of initial value problems of third order ordinary differential equations. In this way, there is no need of providing starting values. The procedure is by converting it to codes in Matlab environment and implement on digital computer.

4.1.1 Test problem 1.

We consider a non homogenous initial value problem of third order ordinary differential equation:

$$y''' + 4y' = x$$
$$y(0) = y'(0) = 0, y''(0) = 1; \quad h = 0.1$$

Whose exact solution is:

$$y(x) = \tfrac{3}{16}(1 - \cos 2x) + \tfrac{1}{18}x^2$$

Our results are as shown in Table 1.

4.1.2 Test problem 2.

We consider a non - linear third order initial value problem:

$$y''' = y'(2xy'' + y');$$
$$y(0) = 1, y'(0) = \tfrac{1}{2}, y''(0) = 0; h = 0.1$$

Whose exact solution is:

$$y(x) = 1 + \tfrac{1}{2}\ln\left\{\frac{2+x}{2-x}\right\}$$

This problem was solved by [5] using block method. Our results are shown in Table 2.

Table 1. Showing results for problem 1

X	Exact solution	Numerical solution	Error
0.1	0.004987516654	0.004987516647	7.36418E - 12
0.2	0.019801063624	0.019801063360	2.61394E - 10
0.3	0.043995722044	0.043995721510	5.28631E - 10
0.4	0.076867491997	0.076867491765	2.31942E - 09
0.5	0.117443317649	0.117443313100	4.51684 E - 09
0.6	0.164557921033	0.164557907500	1.348917E - 08
0.7	0.216881160706	0.216881072300	2.114963E - 08
0.8	0.272974910431	0.272974491900	4.184726E - 07
0.9	0.3313503927549	0.331350249100	1.435985E - 07
1.0	0.3905275318552	0.390527004600	5.271836E - 07

Table 2. Showing results for test problem 2

X	Exact solution	Numerical solution	Error	Error in [5]
0.21	1.1053884478385	1.1053884470631	2.41216E - 12	8.169709E - 11
0.31	1.1562594977993	1.1562594972185	6.38350E - 11	5.122973E - 10
0.41	1.2079463656352	1.2079463650312	3.36475E - 11	2.586994E - 09
0.51	1.2607533165937	1.2607533160124	4.54664E - 11	8.166365E - 09
0.61	1.3150232370960	1.3150232370340	6.24217E - 11	2.142557E - 08
1.71	1.3711532082590	1.3711532071352	5.33622E - 10	4.971783E - 08

4.2 Numerical Results

We make use of the following Notations in the table of results:

X: Value of the independent variable where numerical value is taken.
Exact Solution: Exact result at X value
Numerical Solution: Our Numerical result at X value.
Error: Error of our result at X value.

5. DISCUSSION OF RESULTS

In this paper, we have proposed a fully Implicit Hybrid Block – Predictor Corrector algorithm for the numerical solution of initial value problems of third order ordinary differential equations. We chose step size within the stability interval for better performance of the method. The results of our new method when compared with the block method proposed by [5] showed that our method is more accurate.

4. CONCLUSION

In this paper, we have developed a fully implicit Hybrid Predictor – Corrector method for the numerical integration of the initial value problems of third order ordinary differential equations using the approach of Collocation – interpolation. The approach gave rise to a scheme with continuous variables from where additional schemes (derivatives) were developed; the implementation strategy is by combining the major scheme with its derivatives to form a block. The method was analyzed for its basic properties and it was found to be of high order of accuracy, consistent, zero stable and convergent. The method was then adopted for the solution of initial value sample problems of third order ordinary differential equations. Numerical results compete favourably with that of existing methods. The major contribution to knowledge of this work is that a single step method was shown to be effective and adequate towards solving higher order ODEs directly without the need for the use of separate predictors.

COMPETING INTERESTS

Author has declared that no competing interests exist.

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Peer-review history:
The peer review history for this paper can be accessed here:
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