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gr-connectedness in Topological Spaces

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

We discuss about gr-connectedness and different properties of it, such as remarks, examples and theorems.

Keywords: gr -closed sets; gr -continuous maps; gr -connectedness.

1 Introduction

In topology, Connectedness [1] is a known concept and it is one of the principal topological property corresponding to the intuitive thought of having no breaks, which is used to distinguish topological spaces. Any space is homeomorphic to the topological spaces which are connected. Intermediate Value Theorem, Maximum Value Theorem and Uniform Continuity Theorem are formulated by the properties of connectedness, about continuous functions. Connectedness are to be discussed in digital spaces. Data structure is important in the applications on the computer which relates real topological situation. Connectedness and compactness were studied by

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Whyburn G. T [2], considering the assumption of Hausdorff.The basic properties of connectedness had been discussed through many researchers [3-12]. Mathematicians are motivated through the notions of connectedness to extend the existing concepts.

Stone M. [13] investigated the regular closed sets and properties of gr-closed sets in topological spaces was analyzed by Bhattacharya S. [14]. We plan to study gr-connectedness in this research as it has not been discussed yet even gr-closed set was introduced long years ago.

2 Preliminaries

(Y,τ), (Z,σ) (or simply Y and Z) denotes topological spaces with no separation axioms assumed. For every subset D in (Y,τ), closure of D and interior of D are given by $\mathrm{cl}\,(D)$ and $\mathrm{Int}\,(D)$.

Definition 2.1. A subset D of a topological space (Y,τ) is called gr-closed set [14] when $\mathrm{rcl}(D) \subseteq U$ with $D \subseteq U$ and U is open in (Y,τ).

gr-closed set will be the complement of the gr-open set.

Definition 2.2. A function $h:(Y,\tau)\to(Z,\sigma)$ is called

(i) gr-continuous [15] if the inverse image is gr-closed in (Y,τ) for every closed set in (Z,σ).

(ii) gr-irresolute [15] if the inverse image is gr-closed in (Y,τ) for every gr-closed set in (Z,σ).

Definition 2.3. A topological space Y is T_{gr}-space [15] if every gr-closed subset in Y is closed subset in Y.

3 Concepts in gr-connectedness

Definition 3.1. When Y can not be expressed as a disjoint union of two non-empty gr-open sets, a topological space Y is said to be gr-connected. A subset of Y is gr-connected when it is gr-connected as a subspace.

Example 3.1. *Let $Y=\{a,b\}$ and $\tau=\{Y,\phi,\{a\}\}$. So, (Y,τ) is gr-connected.*

Remark 3.1. Every gr-connected space is connected. But, the following example gives in general, the converse is not true.

Example 3.2. *Let $Y=\{a,b\}$ and $\tau=\{Y,\phi\}$. Clearly, (Y,τ) is connected. As gr-open sets of Y are $\{Y,\phi,\{a\},\{b\}\}$ and $Y=\{a\}\cup\{b\}$ where $\{a\}$ and $\{b\}$ are non-empty gr-open sets, (Y,τ) is not a gr-connected space.*

Theorem 3.3. *In a topological space Y, the following statements are equivalent:*

(i) Each gr-continuous map of Y into a discrete with at least two points is a constant map.

(ii) Only subsets of Y which are both gr-open and gr-closed are Y and ϕ.

(iii) Y is gr-connected.

Proof. $(i)\Rightarrow(ii)$: Let D be both gr-open and gr-closed in Y. Assume that $D\neq\phi$. Let $h:Y\to Z$ be a gr-continuous map defined by $h(D)=\{x\}$ and $h(D^c)=\{y\}$ for some distinct points x and y in Z. Hence by using the assumption h is constant, we can say $D=Y$.

$(ii)\Rightarrow(i)$: Let $h:Y\to Z$ be a gr-continuous map.Hence Y is covered by gr-open and gr-closed covering $\{h^{-1}(z):z\in Z\}$. By the assumption, $h^{-1}(z)=\phi$ or Y for each $z\in Z$. If $h^{-1}(z)=\phi$ for all $z\in Z$, then h

fails to be a map. Therefore, there exists only one point $z \in Z$ such that $h^{-1}(z) = \phi$ and hence $h^{-1}(z) = Y$. This shows that h is a constant map.

$(ii) \Rightarrow (iii)$: Let $Y = C \cup D$ where C and D are disjoint non-empty gr-open subsets of Y. Then, C is both gr-open and gr-closed. By the assumption, $C = \phi$ or Y. Hence Y is gr-connected.

$(iii) \Rightarrow (ii)$: Let D be any gr-open and gr-closed subset of Y. Then, D^c is both gr-open and gr-closed. Then, Y is disjoint union of the gr-open sets D and D^c. Hence from the hypothesis of (iii), $D = \phi$ or $D = Y$.

□

Theorem 3.4. *Z is connected when $h : Y \to Z$ is a gr-continuous, onto and Y is gr-connected.*

Proof. Let Z be not connected and $Z = C \cup D$ where C and D are the disjoint non-empty open set in Z. $Y = h^{-1}(C) \cup h^{-1}(D)$ where $h^{-1}(C)$ and $h^{-1}(D)$ are the disjoint non-empty gr-open sets in Y as h is gr-continuous and onto. But, Y is gr-connected. So, this is a contradiction. Therefore, Z is connected. □

Theorem 3.5. *Z is gr-connected in case of $h : Y \to Z$ is a gr-irresolute surjection and X is gr-connected.*

Proof. Let Y be not gr-connected and $Y = C \cup D$ where C and D are disjoint non-empty gr-open set in Z. $Y = h^{-1}(C) \cup h^{-1}(D)$ where $h^{-1}(C)$ and $h^{-1}(D)$ are disjoint non-empty gr-open sets in Y as h is gr-irresolute and onto. So, this is a contradiction because Y is gr-connected. Therefore, Y is connected. □

Theorem 3.6. *Every gr-connected space is connected.*

Proof. Let Y be a gr-connected space and Y be not connected. So, $Y = C \cup D$ with C and D are disjoint, non-empty open sets. So, Y is not a gr-connected as every open set is gr-open set in Y and this is a contradiction. Therefore, Y is connected. □

The Converse of the above theorem is true whenever Y is a T_{gr}-space.

Theorem 3.7. *Y is connected if and only if it is gr-connected, in case of Y is a T_{gr}-space. .*

Proof. Let Y be connected so that Y can not be expressed as disjoint union of two non-empty open subsets of Y. Assume Y is not a gr-connected space and consider C and D are two gr-open subsets of Y such that $Y = C \cup D$ with $C \cap D = \phi$ and $C \subset Y$, $D \subset Y$. C, D are open subsets of Y as Y is T_{gr}-space and C, D are gr-open. So, this is a contradiction. Therefore, Y is connected. Hence Y is gr-connected. Conversely, let's assume that Y is gr-connected and so Y will not be equal as disjoint union of two non-empty gr-open subsets of Y. Every gr-open subset of Y is open because Y is a T_{gr}-space. Therefore, Y will not be equal as disjoint union of two non-empty open subsets of Y. So, Y is connected. □

Theorem 3.8. *Z lies entirely within C or D when Y can be written as the union of two gr-open sets, C and D of Y, where Y is a topological space and Z is a gr-connected subspace of Y.*

Proof. Let $Y = C \cup D$ and $C \cap D = \phi$. $C \cap Y$ and $D \cap Y$ are gr-open in Y as C and D are gr-open in Y. So, $C \cap Y$ and $D \cap Y$ are disjoint and their union is Y. They form a separation of Y in case of both are non-empty so that it will be a contradiction. Therefore, one of them should be empty. Assume that $C \cap Y = \phi$ and so $Y = (CnY) \cup (DnY) \Rightarrow Y = \phi \cup (DnY) \Rightarrow Y \subset D$. Likewise, we may consider $D \cap Y = \phi$. □

Theorem 3.9. *The gr-closure of gr-connected set is gr-connected.*

Proof. Let D be a gr-connected subset in a topological space (Y,τ). Hence we should prove that $gr\,\mathrm{cl}\,(D)$ is gr-connected. Otherwise, $gr\,\mathrm{cl}\,(D) = P \cup Q$ where P, Q are disjoint gr-open sets . So, $D \subseteq gr\,\mathrm{cl}\,(D) = P \cup Q \Rightarrow D \subseteq P$ or $D \subseteq Q$. $D \subseteq P \Rightarrow gr\,\mathrm{cl}\ D \subseteq gr\,\mathrm{cl}\,P \Rightarrow gr\,\mathrm{cl}\,D \cap Q \subseteq gr\,\mathrm{cl}\,P \cap Q = \phi$.

$\Rightarrow\ gr\,\mathrm{cl}\ D \cap Q$ (1)

Also, $gr\,\mathrm{cl}\,(D) = P \cup Q \Rightarrow Q \subseteq gr\,\mathrm{cl}\,D$ implies $Q \cap gr\,\mathrm{cl}\,D = Q$ (2)

From (1) and (2), $Q = \phi$. This is a contradiction and we can get $P = \phi$ in similar way when $D \subseteq Q$. Therefore, $gr\,\mathrm{cl}\,(D)$ must be gr-connected. □

Theorem 3.10. *The union of any family of gr-connected sets having non-empty intersection property is gr-connected.*

Proof. Let $\{E_\alpha : \alpha \in I\}$ be a family of gr-connected subsets with the property that $\cap\{E_\alpha : \alpha \in I\}$ is nonempty. Let $E = \cup\{E_\alpha : \alpha \in I\}$. So, we should prove that E is gr-connected. Otherwise, E can be written as Union of two non-empty disjoint gr-open sets such that $\cup E_\alpha = E = P \cup Q$ and $E_\alpha \subseteq P \cup Q$, for every α. Since each E_α is connected, $E_\alpha \subseteq P$ or $E_\alpha \subseteq Q$ for each $\alpha \in I \Rightarrow \cup E_\alpha \subseteq P or \cup E_\alpha \subseteq Q$. $\Rightarrow E_\alpha \subseteq P$ or $E_\alpha \subseteq Q$ (*)

Since $\cap\{E_\alpha : \alpha \in I\}$ is non-empty, let $x \in \cap\{E_\alpha : \alpha \in I\}$. So, $x \in E_\alpha$ for every $\alpha \in I$. Therefore, $x \in E = \cup\{E_\alpha : \alpha \in I\}$. Hence $x \in E \Rightarrow x \in P$ or $x \in Q$ [by (*)]. Therefore, x cannot belong to both P and Q. If $x \in P$, then $x \notin Q \Rightarrow E \subsetneqq Q$ [by (*)] $\Rightarrow E \subseteq P$. This is a contradiction and so, E must be gr-connected. □

Theorem 3.11. *If $P \subseteq Q \subseteq gr\ cl(P)$, then Q is also gr-connected, where P is a gr-connected subset of Y.*

Proof. Let P be a gr-connected and therefore, it is required to prove that Q is gr-connected. In contrary, we will assume that Q is disconnected. So, $Q = C \cup D$ with C and D are disjoint gr-open sets. Therefore, P must lie entirely in C or D as $P \subseteq Q$. Let's assume $P \subseteq C$ and so, $gr\,\mathrm{cl}\,(P) \subseteq gr\,\mathrm{cl}(C)$. Further, $gr\,\mathrm{cl}\,(P)$ $\cap D \subseteq gr\,\mathrm{cl}(C)\ \cap D = \phi$. Now, $\phi \subseteq D \subseteq gr\ \ \mathrm{cl}\,(P) \cap D \subseteq \phi$. $\therefore\ D = \phi$. This is a contradiction and so, Q is gr-connected. □

4 Conclusion

gr-connectedness in the topological spaces has been discussed as the new definion by using gr-closed sets and we have proved more theorems for gr-connectedness by considering different conditions. Hence this gr-connectedness concept can be applied to other existing connectedness concepts for generalizing new furthermore concepts. Further, we have decided to use this gr-connectedness concept to formulate other new different topological properties and in different toological spaces.

Competing Interests

Authors have declared that no competing interests exist.

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