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An Asymptotic Method for Over-damped Forced Nonlinear Vibration Systems with Slowly Varying Coefficients

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Authors' contributions

This work was carried out in supervision of author PD. Author PD designed the study, wrote the protocol and supervised the work. All authors performed the mathematical analysis of the study and the simulation and approved the final manuscript.

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Abstract

A technique is presented for obtaining an asymptotic solution of over damped nonlinear forced vibrating systems by general Struble's technique and extended KBM method with varying coefficients. The implementation of the presented method is illustrated by an example. The first order analytical approximate solutions obtained by the method for different initial conditions show a good agreement with those obtains by numerical method.

Keywords: Non-autonomous nonlinear system; over damped vibrating system; varying coefficient; perturbed equation; external force.

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1 Introduction

The asymptotic method Krylov-Bogoliubov-Mitroplshkii (KBM) [1-3] is particularly convenient and extensively used methods to study nonlinear differential systems with small nonlinearities. Originally, the method was developed by Krylov and Bogoliubov [1] for obtaining periodic solution of a second order nonlinear differential equation. Letter, the method was amplified and justified mathematically by Bogoliubov and Mitropolishkii [2,3]. Popov [4] extended the method to a damped oscillatory process in which a strong linear damping force acts. Murty, Dekshatulu and Krisna [5] extended the method to over-damped nonlinear system. Shamsul [6-8] investigated over-damped nonlinear systems and found approximate solutions of *Duffing*'s equation when one root of the unperturbed equation was respectively double or triples of the other. Shamsul [9] has presented a unified method for solving an *n*-th order differential equation (autonomous) characterized by oscillatory, damped oscillatory and non-oscillatory processes with slowly varying coefficient. Pinakee et al. [10] has presented extended KBM method for under-damped, damped and over-damped vibrating systems in which the coefficients change slowly and periodically with time. Pinakee et al. [11] extended the technique for damped forced nonlinear system with varying coefficients. Recently Shamsul [12] has developed the general Struble's techniques for several damping effect. The aim of this paper is to find a solution of over damped nonlinear forced vibrating systems that vary slowly with time in which a external force acts and one of the eigen-values is multiple (more than one hundred times; *i.e*., Centuple) of the other eigen-value and measure better result for strong nonlinearities.

2 Methods

Let us consider a nonlinear non-autonomous differential system governed by

$$\ddot{x} + 2k_1(\tau)\dot{x} + (k_2^2 + k_3\cos\tau)x = -\varepsilon\, f(x, \dot{x}, \tau, \nu t), \qquad \tau = \varepsilon t \tag{1}$$

where the over-dots denote differentiation with respect to *t*, ε is a small parameter, k_1, k_2 and k_3 are constants, $k_2 = \mathrm{O}(\varepsilon) = k_3$, $\tau = \varepsilon t$ is the slowly varying time, $k_1(\tau) \geq 0$, f is a given nonlinear function. We set $\omega^2(\tau) = (k_2^2 + k_3\cos\tau)$, where $\omega(\tau)$ is known as internal frequency and ν is the frequency of the external acting force.

Putting $\varepsilon = 0$ and $\tau = \tau_0$ = constant, in Eq.(1), we obtain the unperturbed solution of (1) in the form

$$x(t,0) = x_{1,0}e^{\lambda_1(\tau_0)t} + x_{-1,0}e^{\lambda_2(\tau_0)t}, \tag{2}$$

Let Eq. (1) has two eigenvalues, $\lambda_1(\tau_0)$ and $\lambda_2(\tau_0)$ are constants, but when $\varepsilon \neq 0$, $\lambda_1(\tau_0)$ and $\lambda_2(\tau_0)$ vary slowly with time. We may consider that $|\lambda_2(\tau_0)| >> |\lambda_1(\tau_0)|$ and $\lambda_1(\tau_0) \cong -\nu$. When $\varepsilon \neq 0$, we seek a solution of Eq. (1) in the form

$$x(t,\varepsilon) = x_1(t,\tau) + x_{-1}(t,\tau) + \varepsilon u_1(x_1, x_{-1}, t, \tau) + \varepsilon^2 u_2(x_1, x_{-1}, t, \tau) + \ldots, \tag{3}$$

where x_1 and x_{-1} satisfy the equations

$$\begin{aligned} \dot{x}_1 &= \lambda_1(\tau)x_1 + \varepsilon X_1(x_1, x_{-1}, \tau) + \varepsilon^2 X_1(x_1, x_{-1}, \tau)\ldots, \\ \dot{x}_{-1} &= \lambda_2(\tau)x_{-1} + \varepsilon X_{-1}(x_1, x_{-1}, \tau) + \varepsilon^2 X_{-1}(x_1, x_{-1}, \tau)\ldots, \end{aligned} \tag{4}$$

Differentiating $x(t,\varepsilon)$ two times with respect to *t*, substituting for the derivatives $\ddot{x}$ and *x* in the original equation (1) and equating the coefficient of ε, we obtain

$$\begin{aligned}&(\lambda_1 x_1 Dx_1 + \lambda_2 x_{-1} Dx_{-1}) X_1 + (\lambda_1 x_1 Dx_1 + \lambda_2 x_{-1} Dx_{-1}) X_{-1} + \lambda_1' x_1 + \lambda_2' x_{-1} - \lambda_2 X_1 - \lambda_1 X_{-1} + \\ &+ (\lambda_1 x_1 Dx_1 + \lambda_2 x_{-1} Dx_{-1} - \lambda_1)(\lambda_1 x_1 Dx_1 + \lambda_2 x_{-1} Dx_{-1} - \lambda_2) u_1 \\ &= -f^{(0)}(x_1, x_{-1}, \tau, \nu t),\end{aligned} \tag{5}$$

where $\lambda_1' = \dfrac{d\lambda_1}{d\tau},\ \lambda_2' = \dfrac{d\lambda_2}{d\tau},\ Dx_1 = \dfrac{\partial}{\partial x_1}, Dx_{-1} = \dfrac{\partial}{\partial x_{-1}},\ f^{(0)} = f(x_0, \dot{x}_0, \tau, \nu t)$

Herein it is assumed that $f^{(0)}$ can be expanded in Taylor's series

$$f^{(0)} = \sum_{r_1, r_2 = 0}^{\infty} F_{r_1, r_2}(\tau) x_1^{r_1} x_{-1}^{r_2} \tag{6}$$

To obtain a special over-damped solution of (1), we impose a restriction that $u_1 \cdots$ exclude the terms $x_1^{i_1} x_{-1}^{i_2}, i_1\lambda_1 + i_2\lambda_2 < (i_1 + i_2)k(\tau_0), \quad i_1, i_2 = 0,1,2\cdots$. The assumption assures that $u_1 \cdots$ are free from secular type terms $te^{-\lambda_1 t}$. This restriction guarantees that the solution always excludes *secular*-type terms or the first harmonic terms, otherwise a sizeable error would occur [11]. Moreover, we assume that X_1 and X_{-1} respectively contains terms x_1 and x_{-1}.

3 Example

As example of the above procedure, let us consider a nonlinear non-autonomous system with slowly varying coefficients

$$\ddot{x} + 2k_1(\tau)\dot{x} + (k_2^2 + k_3 \cos\tau)x = -\varepsilon x^3 + \varepsilon E e^{-(.04)t}, \tag{7}$$

Here over dots denote differentiation with respect to t. $x_0 = x_1 + x_{-1}$ and the function $f^{(0)}$ becomes,

$$f^{(0)} = -(x_1^3 + 3x_1^2 x_{-1} + 3x_1 x_{-1}^2 + x_{-1}^3) + Ee^{-(.04)t}. \tag{8}$$

Following the assumption (discussed in section 2) excludes we substitute in (5) and separate it into two parts as

$$\begin{aligned}&(\lambda_1 x_1 Dx_1 + \lambda_2 x_{-1} Dx_{-1}) X_1 + (\lambda_1 x_1 Dx_1 + \lambda_2 x_{-1} Dx_{-1}) X_{-1} + \lambda_1' x_1 + \lambda_2' x_{-1} - \lambda_2 X_1 - \lambda_1 X_{-1} \\ &= -(x_1^3 + 3x_1^2 x_{-1} + x_{-1}^3) + E^{-(.04)t}\end{aligned} \tag{9}$$

and $$(\lambda_1 x_1 Dx_1 + \lambda_2 x_{-1} Dx_{-1} - \lambda_1)(\lambda_1 x_1 Dx_1 + \lambda_2 x_{-1} Dx_{-1} - \lambda_2) u_1 = -(3x_1 x_{-1}^2) \tag{10}$$

The particular solution of (10) is

$$u_1 = c_1\, x_1 x_{-1}^2 \tag{11}$$

Where $c_1 = -3/2\lambda_1(\lambda_1 + \lambda_2)$

Now we have to solve (9) for two functions X_1 and X_{-1} (discussed in section 2)

The particular solutions are

$$\left(\lambda_1 x_1 D x_1 + \lambda_2 x_{-1} D x_{-1}\right) X_1 + \lambda_1' x_1 - \lambda_2 X_1 = -(x_1^3 + 3x_1^2 x_{-1}) + E e^{-(.04)t} \quad (12)$$

and $$\left(\lambda_1 x_1 D x_1 + \lambda_2 x_{-1} D x_{-1}\right) X_{-1} + \lambda_2' x_{-1} - \lambda_1 X_{-1} = -x_{-1}^3 \quad (13)$$

The particular solution of (12)-(13) is

$$X_1 = \lambda_1' x_1 n_1 + n_2 x_1^3 + n_3 x_1^2 x_{-1} + E n_4, \text{ and } X_{-1} = \lambda_2' x_{-1} l_1 + l_2 x_{-1}^3, \quad (14)$$

where

$$n_1 = -1/(\lambda_1 - \lambda_2),\ n_2 = -1/(3\lambda_1 - \lambda_2),\ n_3 = -3/2\lambda_1, n_4 = 1/(-.04 - \lambda_2)$$
$$l_1 = 1/(\lambda_1 - \lambda_2), \quad l_2 = -1/(3\lambda_2 - \lambda_1)$$

Substituting the functional values of X_1, X_{-1} into (4) and rearranging, we obtain

$$\dot{x}_1 = \lambda_1 x_1 + \varepsilon\left(\lambda_1' x_1 n_1 + n_2 x_1^3 + n_3 x_1^2 x_{-1}\right) + \varepsilon E n_4 \quad (15)$$

and $$\dot{x}_{-1} = \lambda_2 x_{-1} + \varepsilon\left(\lambda_2' x_{-1} l_1 + l_2 x_{-1}^3\right) \quad (16)$$

Therefore, the first order solution of (7) is

$$x(t, \varepsilon) = x_1 + x_{-1} + \varepsilon u_1, \quad (17)$$

4 Results and Discussion

Asymptotic solution of over damped forced nonlinear vibrating system is obtained based on the general Struble's technique and extended KBM method with slowly varying coefficients. The solution has been determined under the technique which gives better result for long time. In order to test the accuracy of an approximate solutions obtain by a perturbation method, we compare the approximate solution to the numerical solution (consider to be exact). With regard to such a comparison concerning the presented general Struble's technique and extended KBM method of this paper, we refer to the works of Murty, Dekshatulu and Krishna [5] Shamsul [6-9] and Pinakee et al. [10-11]. In this paper we have compared the perturbation solution (17) to those obtained by Runge-Kutta (Fourth order) method.

First of all, x is calculated by (17) with initial conditions $[x(0) = 1.0000 \quad \dot{x} = 0.0000]$ or $x_1 = 1.0000$, $x_{-1} = -.000128$ for $\varepsilon = .1$, $\omega = \omega_0\sqrt{(k_2^2 + k_3 \cos\tau)}$, $\lambda_1 = -.04$, $\lambda_2 = -6$. The solutions are various values of t are presented in the second column of Table 1. The corresponding numerical solutions is computed by Runge-Kutta fourth-order method and are given in the third column of the Table 1. All the results are shown in Table 1. Percentage errors have also been calculated and given in the fourth column of the Table 1.

Secondly, we have computed by (17) for another sets of initial conditions (i) $[x(0)=1.0000$ $\dot{x}=0.0000$] or $x_1=1.0000$, $x_{-1}=-..000075$ for $\varepsilon=.9$, $\omega=\omega_0\sqrt{(k_2^2+k_3\cos\tau)}$, $\lambda_1=-.04$, $\lambda_2=-6$ and (ii) $[x(0)=1.0000$ $\dot{x}=0.0000$] or $x_1=1.0000$, $x_{-1}=-.000336$ for $\varepsilon=1.0$, $\omega=\omega_0\sqrt{(k_2^2+k_3\cos\tau)}$, $\lambda_1=-.04$, $\lambda_2=-6$. The solutions are various values of t are presented in the second column of Tables 2 and 3. The corresponding numerical solutions are also computed by Runge-Kutta fourth-order method and are given in the third column of the Tables 2 and 3. Percentage errors have also been calculated and given in the fourth column of the Tables 2 and 3.

Table 1.

t	x_{nu}	x_{va}	$E(\%)$
0.0	1	1	0
1.0	.9999056	.998789	.026732
2.0	.996139	.995594	.054741
3.0	.99145	.990667	.079038
5.0	.97765	.976499	.11787
7.0	.959208	.957806	.146376
9.0	.937342	.935778	.167134
10.0	.925431	.923811	.175361
20.0	.789022	.787384	.208031
30.0	.649285	.647974	.202323
40.0	.522599	.521626	.186532

Initial conditions $x(0)=1.0000$ $\dot{x}=0,0000$ *or* $x_1=1.0000$, $x_{-1}=-.000128$ *for* $\lambda_1=-.04$, $\lambda_2=-6$, $\omega=\omega_0\sqrt{(k_2^2+k_3\cos\tau)}$, $\varepsilon=0.1$

Table 2.

t	x_{nu}	x_{va}	$E(\%)$
0.0	1	1	0
1.0	.998714	.996695	.202569
2.0	.992102	.988839	.329983
3.0	.98217	.978387	.386657
5.0	.957753	.95378	.416553
7.0	.931237	.927391	.414712
9.0	.90445	.900771	.408428
10.0	.89116	.88756	.405606
20.0	.765808	.76286	.386441
30.0	.654577	.52165	.369845
40.0	.555996	.554052	.35087

Initial conditions $x(0)=1.0000$ $\dot{x}=0.0000$ *or* $x_1=1.0000$, $x_{-1}=-.000075$ *for* $\lambda_1=-.04$, $\lambda_2=-6$, $\omega=\omega_0\sqrt{(k_2^2+k_3\cos\tau)}$, $\varepsilon=0.9$

Finally, we have computed by (16) for another sets of initial conditions (i) $[x(0)=1.0000 \quad \dot{x}=0.0000]$ or $x_1=1.0000, \quad x_{-1}=-.002471$ for $\varepsilon=1.2,\ \omega=\omega_0\sqrt{(k_2^2+k_3\cos\tau)}, \quad \lambda_1=-.04, \quad \lambda_2=-6$ and (ii) $[x(0)=1.0000 \quad \dot{x}=0.0000]$ or $x_1=1.0000, \quad x_{-1}=-.001525$ for $\varepsilon=1.3,\ \omega=\omega_0\sqrt{(k_2^2+k_3\cos\tau)}, \quad \lambda_1=-.04, \quad \lambda_2=-6$ (iii) $[x(0)=1.0000 \quad \dot{x}=0.0000]$ or $x_1=1.0000, \quad x_{-1}=-.003236$ for $\varepsilon=1.4,\ \omega=\omega_0\sqrt{(k_2^2+k_3\cos\tau)}, \quad \lambda_1=-.04, \quad \lambda_2=-6$. The solutions are various values of t are presented in the second column of Tables 4, 5 and 6. The corresponding numerical solutions are also computed by Runge-Kutta fourth-order method and are given in the third column of the Tables 4, 5 and 6. Percentage errors have also been calculated and given in the fourth column of the Tables 4, 5 and 6.

Table 3.

t	x_{nu}	x_{va}	$E(\%)$
0.0	1	1	0
1.0	.998451	.996268	.219118
2.0	.991337	.9879	.34791
3.0	.980932	.977026	.399785
5.0	.955964	.951949	.421766
7.0	.929323	.925463	.417089
9.0	.902635	.898944	.410593
10.0	.884438	.885824	.407982
20.0	.765372	.762391	391007
30.0	.655416	.652963	.375672
40.0	.557977	.555989	.357561

Initial conditions $x(0)=1.0000 \ \ \dot{x}=0.0000$ *or* $x_1=1.0000, \quad x_{-1}=-.000336$ *for* $\lambda_1=-.04, \quad \lambda_2=-6,$
$\omega=\omega_0\sqrt{(k_2^2+k_3\cos\tau)}, \quad \varepsilon=1.0$

Table 4.

t	x_{nu}	x_{va}	$E(\%)$
0.0	1	1	0
1.0	.996698	.994302	.240973
2.0	.988093	.984475	.367506
3.0	.976649	.972652	.410938
5.0	.950764	.946752	.423765
7.0	.924066	.920215	.418489
9.0	.897667	.893969	.413661
10.0	.884669	.881042	.411672
20.0	.762939	.75991	.3986
30.0	.655212	.6527	.384863
40.0	.559789	.557734	.368455

Initial conditions $x(0)=1.0000 \ \ \dot{x}=0.0000$ *or* $x_1=1.0000, \quad x_{-1}=-.002471$ *for* $\lambda_1=-.04, \quad \lambda_2=-6,$
$\omega=\omega_0\sqrt{(k_2^2+k_3\cos\tau)}, \quad \varepsilon=1.2$

Table 5.

t	x_{nu}	x_{va}	$E(\%)$
0.0	1	1	0
1.0	.997394	.994798	.260958
2.0	.988831	.985024	.386488
3.0	.977267	.973149	.423162
5.0	.951233	.947181	.427796
7.0	.924553	.920679	.420776
9.0	.898248	.894529	.41575
10.0	.88531	.88166	.413992
20.0	.76424	.761181	.401876
30.0	.657153	.654607	.388936
40.0	.562324	.560233	.373238

Initial conditions $x(0)=1.0000$ $\dot{x}=0.0000$ *or* $x_1=1.0000,$ $x_{-1}=-.001525$ *for* $\lambda_1=-.04,$ $\lambda_2=-6$, $\omega=\omega_0\sqrt{(k_2^2+k_3\cos\tau)},$ $\varepsilon=1.3$

Table 6.

t	x_{nu}	x_{va}	$E(\%)$
0.0	1	1	0
1.0	.996061	.993422	.265647
2.0	.98667	.982855	.388155
3.0	.974662	.970565	.422125
5.0	.948367	.944339	.426542
7.0	.921756	.917892	.420965
9.0	.895614	.891894	.41709
10.0	.882767	.879114	.415532
20.0	.762641	.759567	.404704
30.0	.656433	.653868	.392281
40.0	.562408	.560294	.377302

Initial conditions $x(0)=1.0000$ $\dot{x}=0.0000$ *or* $x_1=1.0000,$ $x_{-1}=-.003263$ *for* $\lambda_1=-.04,$ $\lambda_2=-6$, $\omega=\omega_0\sqrt{(k_2^2+k_3\cos\tau)},$ $\varepsilon=1.4$

From Tables 1, 2, 3, 4, 5 and 6, it is clear that percentage errors are smaller than 1% and thus (17) show a good coincidence with the numerical solution.

5 Conclusion

In this article a technique is developed for obtaining the solution of nonlinear non autonomous vibrating systems based on the general Struble's technique and extended KBM method with slowly varying coefficients under the action of external forces. The solutions agree nicely with the numerical solutions when one of the eigen-values is multiple (more than one hundred times; *i.e.*, Centuple) of the other eigen-value and measure better result for strong nonlinearities.

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Competing Interests

Authors have declared that no competing interests exist.

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