An Elementary Approach to the Option Pricing Problem

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

Review Article

Abstract

Our goal here is to discuss the pricing problem of European and American options in discrete time using elementary calculus so as to be an easy reference for first year undergraduate students. Using the binomial model we compute the fair price of European and American options. We explain the notion of Arbitrage and the notion of the fair price of an option using common sense. We give a criterion that the holder can use to decide when it is appropriate to exercise the option. We prove the put-call parity formulas for both European and American options and we discuss the relation between American and European options. We give also the bounds for European and American options. We also discuss the portfolio optimization problem and the fair value in the case where the holder can not produce the opposite portfolio.

1 Introduction

Our starting point was the paper [1] in which the authors introduce the binomial model and explain how one can use it to evaluate the fair price of a European option. Our goal here is to study the option pricing problem in discrete time using the binomial method and basic calculus so as to be an easy reference for first year undergraduate students.

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There are many books that discuss the binomial model in a more advanced setting, see for example [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20]. Our aim here is to explain the binomial method using elementary calculus but not losing any of the mathematical accuracy.

We begin our discussion from the beginning, i.e. we describe firstly how one can model the movement of an asset. Then, we describe how can someone construct portfolios with prescribed final and intermediate values and we discuss both the European and American type options. We also discuss the notion of the Arbitrage and prove that the binomial model does not admit Arbitrage under some suitable condition. We prove the put-call parity formulas for both European and American options and we discuss the relation between American and European options. We give also the bounds for European and American options. We also discuss the portfolio optimization problem and the fair value in the case where the holder can not produce the opposite portfolio.

Suppose that our market consists of one risky asset, say S, and one non-risky, say B, with daily interest rate r. We consider for simplicity that we have only one period, time zero and time one.

In time zero no one knows the value of the risky asset in time one, i.e. no one knows S_1. How can we model this? We can for example, study the way that the asset behaved the last, say, one month and denote the average of the percentage of got up as u and the average of percentage of got down as d. Then we can suppose that the risky asset will follow the same path in the future and thus we can write schematically

$n = 0$ $\qquad\qquad\qquad\qquad$ $n = 1$
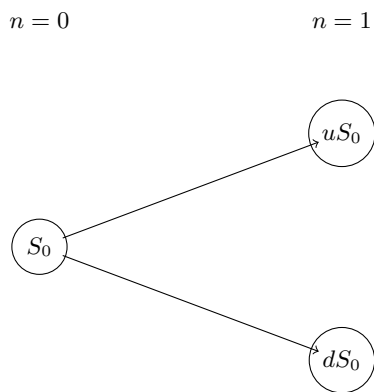
2   Constructing a Portfolio with Prescribed Final Values

At time zero, someone can buy a shares of the risky asset and put the amount b in the bank therefore constructing a portfolio with initial value
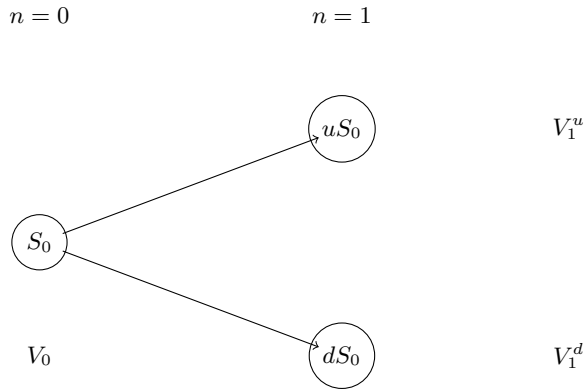
$$V_0 = aS_0 + b$$

If our time period is one day then after one day the value of the portfolio will be

$$V_1^u = a(uS_0) + b(1 + r)$$

if the value of the asset will go up, and

$$V_1^d = a(dS_0) + b(1 + r)$$

if the value of the asset will go down. We can write it schematically

$n = 0$ $n = 1$

Suppose now that we are given specific numbers A, B and we are asked to construct a portfolio (a, b) such that, under the above hypotheses, will have final values $V_1^u = A$ and $V_1^d = B$. How much money b we will have to put in the bank at time zero and how many shares of the risky asset should we buy at time zero? Schematically we have the following

$n = 0$ $n = 1$
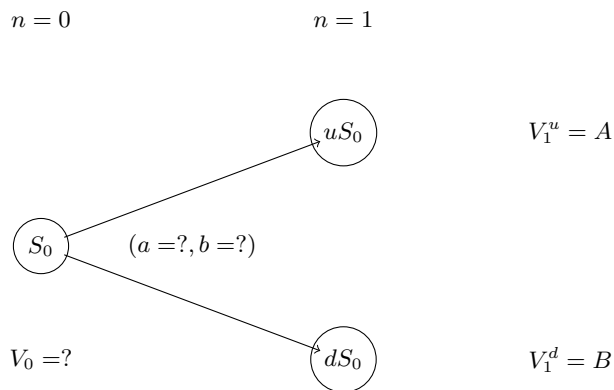
We have to solve two equations with two unknowns

$$a(uS_0) + b(1 + r) = A,$$
$$a(dS_0) + b(1 + r) = B$$

The determinant of this system will be non zero if $S_0 \neq 0$, $r \neq -1$ and $u > d$. Under the above hypotheses we have that the solution of the system will be

$$a = \frac{A - B}{(u - d)S_0},$$
$$b = \frac{Bu - Ad}{(u - d)(1 + r)}$$

Therefore the initial value of our portfolio has to be

$$V_0 = aS_0 + b = \frac{1}{1+r}\big(qA + (1-q)B\big)$$

where $q = \frac{1+r-d}{u-d}$ and this results just if one replaces a, b from above to the

$$V_0 = aS_0 + b$$

If $b \leq 0$ then that means that we have to borrow money from the bank and if $a \leq 0$ means that we have to sell a shares of the asset that we do not belong.

So, with the amount V_0 we have constructed a portfolio (a, b) with final values A, B. Is there any chance to construct a portfolio with final values $A + \varepsilon_1$ and $B + \varepsilon_2$ (with $\varepsilon_1 >$ and $\varepsilon_2 > 0$) and initial value V_0? Let us find first the portfolio (a_1, b_1) with final values $A + \varepsilon_1$ and $B + \varepsilon_2$

$$
\begin{aligned}
a_1 &= a + \frac{\varepsilon_1 - \varepsilon_2}{(u-d)S_0}, \\
b_1 &= b + \frac{\varepsilon_2 u - \varepsilon_1 d}{(u-d)(1+r)}
\end{aligned}
$$

Note that the portfolio (a, b) have initial value V_0 and we want also portfolio (a_1, b_1) to have the same initial value but bigger final values. Therefore, it must holds

$$\frac{\varepsilon_1 - \varepsilon_2}{(u-d)} + \frac{\varepsilon_2 u - \varepsilon_1 d}{(u-d)(1+r)} = 0$$

In other words, it must holds

$$\varepsilon_1(1 + r - d) + \varepsilon_2(u - (1+r)) = 0 \tag{2.1}$$

3 Arbitrage and Smallest Initial Value

Is there any portfolio (a, b) with initial value $V_0 = 0$ and final values

$$
\begin{aligned}
V_1^u &> 0 \\
V_1^d &\geq 0
\end{aligned}
$$

or

$$
\begin{aligned}
V_1^u &\geq 0 \\
V_1^d &> 0
\end{aligned}
$$

If someone can construct such portfolios then he can borrow/put b money from/to the bank to buy/sell a shares of the asset (again and again) and at the end he makes profit with zero initial capital and without any risk. Of course in real world there are not such portfolios so in our mathematical model we should exclude such a situation which we call Arbitrage.

Theorem 3.1. *The binomial model does not admit Arbitrage iff* $0 < d < 1 + r < u$

Proof. Let us suppose that $0 < d < 1 + r < u$ holds. We construct a portfolio (a, b) such that

$$V_0 = aS_0 + b = 0$$

so that $b = -aS_0$. Suppose now that

$$
\begin{aligned}
V_1^u &= a(uS_0) + b(1+r) > 0 \\
V_1^d &= a(dS_0) + b(1+r) \geq 0
\end{aligned}
$$

We will see now that in fact we have $a > 0$. We can write

$$
V_1^u = auS_0 - aS_0(1+r) > 0
$$

Therefore we arrive at

$$
aS_0(u - (1+r)) > 0
$$

Using the fact that $u > 1 + r$ we obtain that $a > 0$. Substituting the equality $b = -aS_0$ in the inequality $V_1^d \geq 0$ we conclude that $d \geq 1+r$ but this is a contradiction. Using the same arguments one concludes that it can not happen (if $d < 1 + r < u$)

$$
\begin{aligned}
V_1^u &\geq 0 \\
V_1^d &> 0
\end{aligned}
$$

Conversely, suppose that the binomial model do not admit Arbitrage. Consider all the possible portfolios with initial value

$$
V_0 = aS_0 + b = 0
$$

If $V_1^u > 0$ then $V_1^d < 0$ otherwise (a, b) is an Arbitrage opportunity.
By summing the inequalities $V_1^u > 0$ and $-V_1^d \geq 0$ we get that $a > 0$. Using these inequalities and that $a > 0$ we get the desired inequality, i.e. $d < 1 + r < u$.

If $V_1^u < 0$ then $V_1^d > 0$ otherwise $(-a, -b)$ is an Arbitrage opportunity. The same arguments drives us to the same conclusion.

If $V_1^u = 0$ then also $V_1^d = 0$ otherwise (a, b) or $(-a, -b)$ is an Arbitrage opportunity. By these two equalities we conclude that $d = 1 + r = u$ and that mean that asset's value remain constant.

From now on we will suppose that $0 < d < 1 + r < u$ in order to avoid Arbitrage in our model.

We have shown that for any A, B one can construct a portfolio (a, b) with final values $V_1^u = A$ and $V_1^d = B$ under the hypotheses that $u > d$, $r \neq -1$ and $S_0 \neq 0$. This is called completeness of the model. What about the smallest initial value of the portfolio with final values A, B? We have shown that if someone want to construct a portfolio with final values $A + \varepsilon_1$ and $B + \varepsilon_2$ then $\varepsilon_1, \varepsilon_2$ should satisfy equation (1). Assuming that our model do not admit Arbitrage, then equation (1) holds iff $\varepsilon_1 = \varepsilon_2 = 0$. Therefore, V_0 is the smallest initial value for our portfolio with final values A, B if our model do not admit Arbitrage.

4 Two Period Binomial Model

We can extend our results for a two period binomial model, i.e.
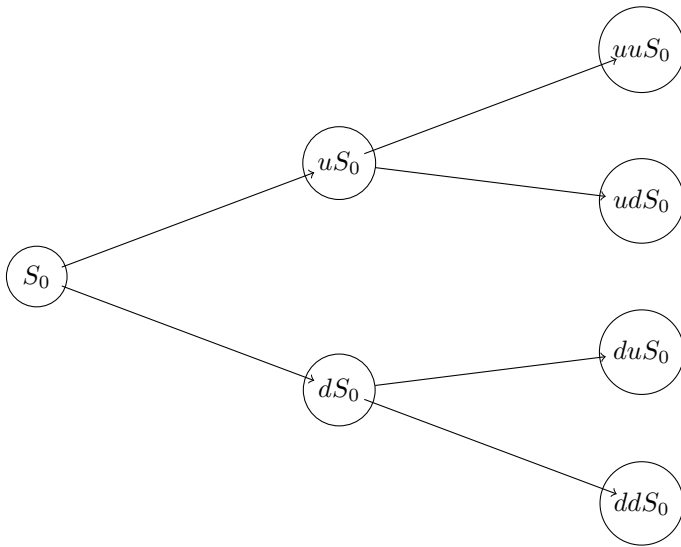
If we do not have Arbitrage for the one period binomial model then the same holds for the two period (and so on) binomial model. We can also construct a portfolio and schematically have the following
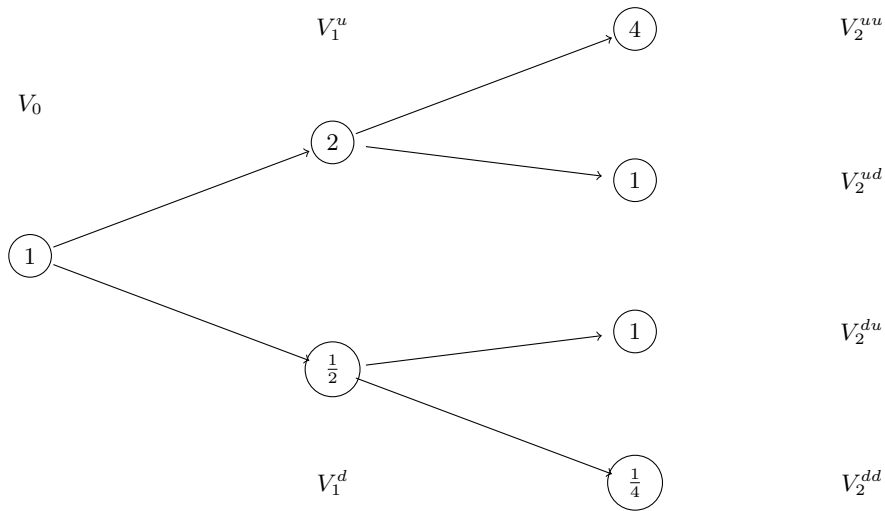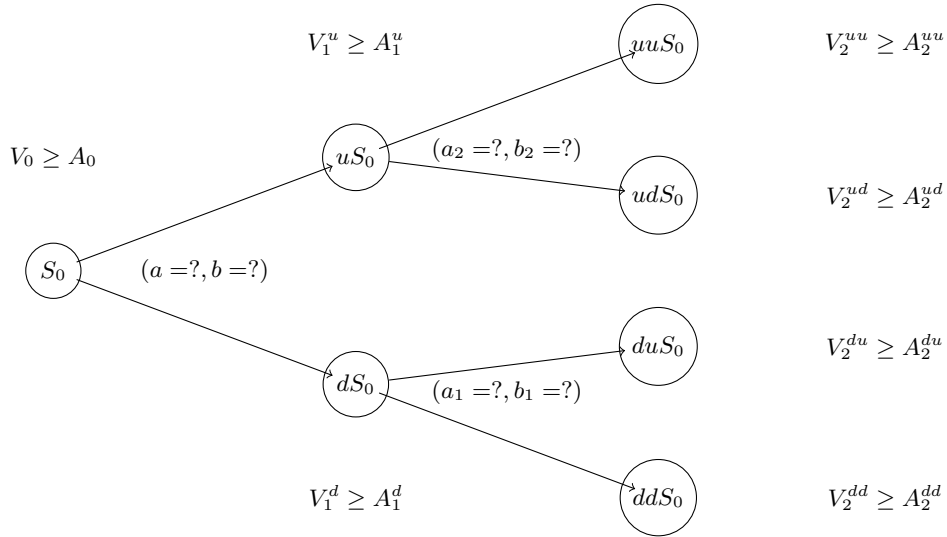

with

$$
\begin{aligned}
V_2^{uu} &= a(uuS_0) + b(1+r)^2, \\
V_2^{ud} &= a(udS_0) + b(1+r)^2, \\
V_2^{du} &= a(duS_0) + b(1+r)^2, \\
V_2^{dd} &= a(ddS_0) + b(1+r)^2
\end{aligned}
$$

Suppose now that we are given specified numbers $A_2^{uu}, A_2^{ud}, A_2^{du}, A_2^{dd}, A_1^u, A_1^d, A_0$ and we are asked

to construct the smallest portfolio such that

$$
\begin{aligned}
V_2^{uu} &\geq A_2^{uu}, \\
V_2^{ud} &\geq A_2^{ud}, \\
V_2^{du} &\geq A_2^{du}, \\
V_2^{dd} &\geq A_2^{dd}, \\
V_1^{u} &\geq A_1^{u}, \\
V_1^{d} &\geq A_1^{d}, \\
V_0 &\geq A_0
\end{aligned}
$$

Schematically we have

How can we construct such a portfolio? We want $V_1^u \geq A_1^u$ and also be such that at time 2 has the values A_2^{uu}, A_2^{ud}. Choosing,

$$
V_1^u = \max\{A_1^u, \frac{1}{1+r}\big(qA_2^{uu} + (1-q)A_2^{ud}\big)\}
$$

we obtain the desired result. The same holds for V_1^d, i.e.

$$
V_1^d = \max\{A_1^d, \frac{1}{1+r}\big(qA_2^{du} + (1-q)A_2^{dd}\big)\}
$$

Then, we choose $V_0 = \max\{A_0, \frac{1}{1+r}\big(qV_1^u + (1-q)V_1^d\big)\}$. Next, we construct portfolios (a, b), (a_2, b_2) and (a_1, b_1).

It is clear that the initial value V_0 is the smallest value to construct a portfolio with the specified requirements.

5 European and American Options

A contract called European call option gives its holder the right but not the obligation to purchase from the writer a prescribed asset S for a prescribed price K at a prescribed time T in the future.

So, at time T the holder of the option has the profit (if any) $(S_T - K)^+$ and that is the amount of money that the writer of the option has to pay at the expiration of the contract. It is obvious that this kind of contract should have an initial cost for the holder. What is the fair value V_0 of this contract? The writer, at time T, using the amount V_0 should construct the profit of the holder $(S_T - K)^+$. Therefore, if we consider our model (one period model for example) the problem is to find V_0, a, b to construct a portfolio with specified final values. Schematically we have the following,

$$V_1^u = (uS_0 - K)^+ = 4/3$$

$(a = ?, b = ?)$

$V_0 = ?$

$$V_1^d = (dS_0 - K)^+ = 0$$
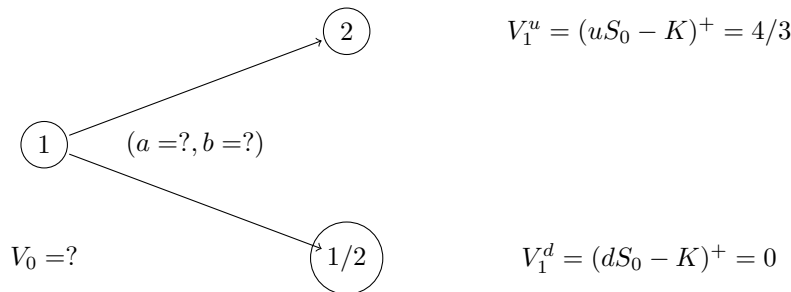
with $S_0 = 1$, $u = 2$ and $d = 1/2$, $K = 2/3$ for example. Therefore the problem can be solved as we have described before. We have seen that if $d < 1 + r < u$ then V_0, computed in this way, is the smallest amount of money that the writer needs for this contract in order to construct a portfolio that eliminates the risk. Note that there is no path in which the writer loses or earn money selling this contract. Moreover, the holder can lose money but also can earn money from this contract. Any price above V_0 will make sure profit (without risk) to the writer. What about the case where the holder of the option do not exercise it even in the case he has positive profit? Then this profit remain to the writer. Is this an Arbitrage? In order to decide if it is an Arbitrage or not we count only all the possible paths of the asset. If for all possible paths of the asset the holder can exercise in a way that the writer will have no sure profit then we do not have Arbitrage. All these results can be extended to two period models and in general to n period models.

A contract called American call option gives its holder the right but not the obligation to purchase from the writer a prescribed asset for a prescribed price at any time until the expiration date T. Therefore, the profit (if any) of the holder is $(S_t - K)^+$ where $t \leq T$ is the exercise time. The problem again is what is the fair price V_0 of this contract. The writer should have enough money for all the circumstances. Considering a two period model the holder can exercise the option at times 0,1 as well at time 2. Therefore, the writer should construct a portfolio which has value at any time at least $(S_t - K)^+$ i.e. the holder's profit (in order to eliminate the risk). So, the problem is to specify the numbers $A_2^{uu}, A_2^{ud}, A_2^{du}, A_2^{dd}, A_1^u, A_1^d, A_0$ and construct a portfolio such that
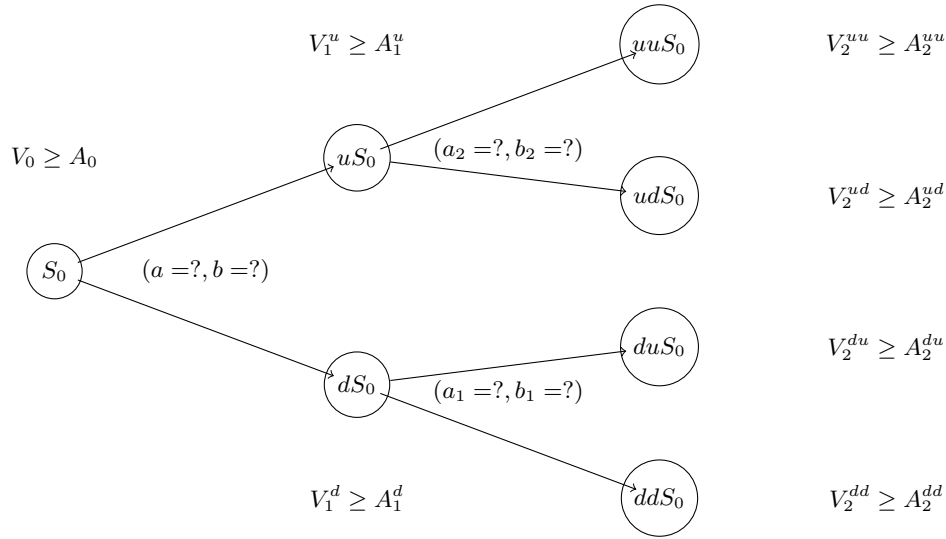
Therefore we calculate V_0 as we have described and also (a, b), (a_1, b_1) and (a_2, b_2). Using this amount of money the writer will be sure that he will have enough money for all possible paths of the asset (i.e. he can construct a portfolio that eliminates the risk). There is no path that the writer will make sure profit, because in each path the holder can exercise in that a way that his profit equals portfolio's value.

Denoting by X_n holder's profit at time n when is the best time to exercise the option? If the holder exercise at time n then the profit will be in fact (considering the amount V_0 that he had paid for this option) $X_n - V_0(1 + r)^n$. Therefore, a criterion is that the holder will exercise when $X_n > V_0(1 + r)^n$, i.e. in this case the holder will earn more money than the case where he puts V_0 at time zero to a bank. If the holder of the option choose to exercise it when $X_n < V_n$ then the writer has a profit. Is this an Arbitrage? No, it is not an Arbitrage, because the notion of the Arbitrage is independent of the choices of the holder. It depends only on all the possible paths of the asset.

As an example consider an American call option with strike price $K = 5/2$, $N = 3$, $u = 2$, $d = 1/2$ and $r = 1/2$. Suppose that the asset moves in the path uu for the first two periods. The holder should decide if he exercise at time $n = 2$ the option or not. Making the calculations the holder should choose to exercise at $n = 2$ because his profit is

$$(S_2^{uu} - K) - V_0(1 + r)^2 = 3/2 - 0.48(1 + 1/2)^2 > 0$$

Note also that the value of the option at that time is $H_2^{uu} = V_2^{uu} = 2.44 > X_2^{uu} = 3/2$ and therefore the writer has a positive profit as well. Of course the holder can choose to wait and if the asset go up again he will make a larger profit but if the asset go down then he loses all the money. If the holder of the option can sell the option or he is able to sell a number of assets that he do not own at time $n = 2$ then he should decide (at this time) what is preferable for him. In any case, at time $n = 2$ he should take a decision about his next move.

We should point out that if for some n we have $V_n^A = X_n$ and suppose further that the holder does not exercise at that time then the writer has more money than he really needs to go to the next step. Therefore, he can consume this extra amount of money and invest the rest of them appropriately in order to eliminate the risk. In this case let us denote the value of the portfolio as V_n^{AC}. He also
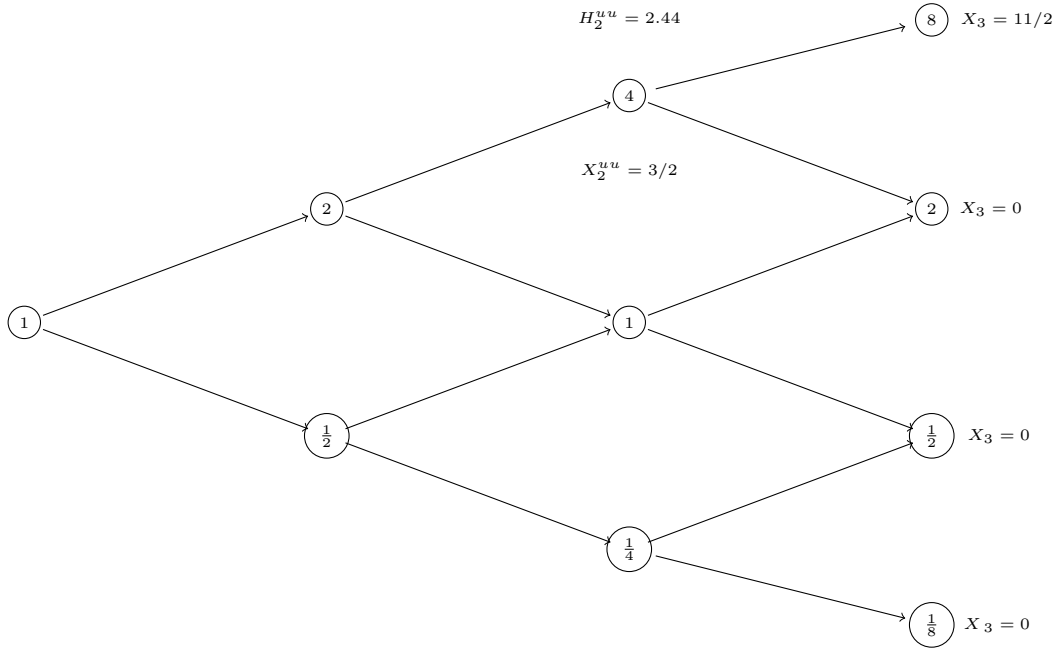
can put this amount of money to the bank or invest it on shares. In this case we denote the value of the portfolio as V_n^A and therefore it holds that

$$V_n^{AC} \leq V_n^A$$

Denote by H_n the following sequence

$$H_n = \begin{cases} X_N, & \text{for } n = N, \\ \max\{X_n, \frac{1}{1+r}\left(qH_{n+1}^u + (1-q)H_{n+1}^d\right)\}, & \text{for } n = N-1, ..., 0 \end{cases}$$

We say that H_n is the fair value of the American option at time n. Note that $V_n^A \geq H_n$.

6 Put-Call Parity Formulas, Relations between European and American Options and Bounds for Options

6.1 European put - call parity

Consider a European call option with strike price K and the corresponding European put option. Let us denote by $V_n^{E,call}$ and $V_n^{E,put}$ the prices of the options working on an N-period binomial model. The following formula holds,

$$V_n^{E,call} - V_n^{E,put} = S_n - K\frac{1}{(1+r)^{N-n}}, \quad n = 0, ..., N$$

Indeed, for $n = N$ we have

$$V_N^{E,call} - V_N^{E,put} = (S_N - K)^+ - (K - S_N)^+ = (S_N - K)^+ - (S_N - K)^- = S_N - K.$$

Suppose that the formula holds for some n. We will show that it holds also for $n-1$,

$$
\begin{aligned}
V_{n-1}^{E,call} - V_{n-1}^{E,put} &= \frac{1}{1+r}\left(q(V_n^{u,E,call} - V_n^{u,E,put}) + (1-q)(V_n^{d,E,call} - V_n^{d,E,put})\right)\\
&= \frac{1}{1+r}\left(q(uS_{n-1} - K\frac{1}{(1+r)^{N-n}}) + (1-q)(dS_{n-1} - K\frac{1}{(1+r)^{N-n}})\right)\\
&= S_{n-1} - K\frac{1}{(1+r)^{N-n+1}}
\end{aligned}
$$

6.2   Relation between European and American options

In general, it is easy to see that

$$V_n^E \le H_n, \quad n = 0,...N$$

Indeed, for $n = N$ we have

$$V_N^E = X_N = H_N$$

where X_N is the holder's profit at time N. Suppose that it holds for some n, that is

$$V_n^E \le H_n$$

We will show that it holds also for $n-1$. We can write

$$
\begin{aligned}
H_{n-1} &= \max\{X_{n-1}, \frac{1}{1+r}\left(qH_n^u + (1-q)H_n^d\right)\}\\
&\ge \max\{X_{n-1}, \frac{1}{1+r}\left(qV_n^{u,E} + (1-q)V_n^{d,E}\right)\}\\
&= \max\{X_{n-1}, V_{n-1}^E\}\\
&\ge V_{n-1}^E
\end{aligned}
$$

Consider now the case where we have a European option with strike price K and the corresponding American option in a N-period model. If we speak about call options and $r \ge 0$ then we will prove that

$$V_n^{E,call} = H_n^{call} \quad \text{(when } r \ge 0), \quad n = 0,...,N$$

We need first to prove that

$$X_n \le V_n^{E,call}, \quad n = 0,...,N$$

We will prove this by induction. For $n = N$ it is obvious, we suppose that it holds for some n and we will prove that it also holds for $n-1$. To do so we work as follows

$$
\begin{aligned}
S_{n-1} - K &= \frac{(qu + (1-q)d)}{1+r}(S_{n-1} - K)\\
&= \frac{1}{1+r}\left(q(uS_{n-1} - K) + (1-q)(dS_{n-1} - K)\right)\\
&\quad + \frac{1}{1+r}(qK(1-u) + (1-q)K(1-d))\\
&\le \frac{1}{1+r}(q(uS_{n-1} - K)^+ + (1-q)(dS_{n-1} - K)^+)\\
&\quad + \frac{1}{1+r}(qK(1-u) + (1-q)K(1-d))\\
&\le \frac{1}{1+r}(qV_n^{u,E,call} + (1-q)V_n^{d,E,call})\\
&= V_{n-1}^{E,call}
\end{aligned}
$$

where we have used the fact that $1 + r = qu + (1 - q)d$ and that

$$qK(1 - u) + (1 - q)K(1 - d) = K - K(1 + r) \leq 0$$

Because $V_{n-1}^{E,call} \geq 0$ we have also that

$$X_{n-1} \leq V_{n-1}^{E,call}, \quad n = 1, ..., N$$

Now we are ready to prove that $V_n^{E,call} = H_n^{call}$ by induction. For $n = N$ is obvious so we suppose that it holds for some n and we will prove that it holds also for $n - 1$.

Indeed,

$$
\begin{aligned}
H_{n-1}^{call} &= \max\{X_{n-1}, \frac{1}{1+r}\left(qH_n^{u,call} + (1-q)H_n^{d,call}\right)\} \\
&= \max\{X_{n-1}, \frac{1}{1+r}\left(qV_n^{u,E,call} + (1-q)V_n^{d,E,call}\right)\} \\
&= \max\{X_{n-1}, V_{n-1}^{E,call}\} \\
&= V_{n-1}^{E,call}
\end{aligned}
$$

Therefore, at any time n there are no extra money for the writer to consume and thus

$$V_n^{A,call} = V_n^{AC,call}$$

Furthermore, if we speak about put options and $r = 0$ then we also have that

$$V_n^{E,put} = H_n^{put} \quad \text{(when } r = 0\text{)}, \quad n = 0, ..., N$$

We will first prove that $V_n^{E,put} \geq X_n$ for $n = 0, ..., N$. For $n = N$ it is obvious so we assume that it holds for some n and we will prove it also for $n - 1$. We work as follows

$$
\begin{aligned}
K - S_{n-1} &= K - dS_{n-1} + S_{n-1}(d - 1) \\
&= (1 - q)(K - dS_{n-1}) + q(K - dS_{n-1}) + S_{n-1}(d - 1) \\
&= (1 - q)(K - dS_{n-1}) + q(K - uS_{n-1}) \\
&\quad + qS_{n-1}(u - d) + S_{n-1}(d - 1) \\
&\leq (1 - q)(K - dS_{n-1})^+ + q(K - uS_{n-1})^+ \\
&\quad + qS_{n-1}(u - d) + S_{n-1}(d - 1) \\
&\leq V_{n-1}^{E,put}
\end{aligned}
$$

where we have used the fact that $q = \frac{1-d}{u-d}$ so that

$$S_{n-1}(d - 1) = -qS_{n-1}(u - d)$$

Note that, $V_{n-1}^{E,put} \geq 0$ therefore we also have

$$X_{n-1} \leq V_{n-1}^{E,put}, \quad n = 1, ..., N - 1.$$

Now, by induction it is easy to prove that $H_n^{put} = V_n^{E,put}$. Indeed, for $n = N$ it is obvious and if it holds for some n then we will prove that it also holds for $n - 1$. Therefore

$$
\begin{aligned}
H_{n-1}^{put} &= \max\{X_{n-1}, \left(qH_n^{u,put} + (1-q)H_n^{d,put}\right)\} \\
&= \max\{X_{n-1}, \left(qV_n^{u,E,put} + (1-q)V_n^{d,E,put}\right)\} \\
&= \max\{X_{n-1}, V_{n-1}^{E,put}\} \\
&= V_{n-1}^{E,put}
\end{aligned}
$$

Finally, since there are no extra money at any time n then

$$V_n^{AC,put} = V_n^{A,put}, \quad \text{(when } r = 0\text{)}.$$

6.3 American put - call parity

Consider now an American call option with strike price K and the corresponding put option. The following inequality holds, for a N-period binomial model, when $r \geq 0$,

$$H_n^{call} - H_n^{put} \leq S_n - K\frac{1}{(1+r)^{N-n}}, \quad n = 0, ..., N$$

To show this we note that $H_n^{call} = V_n^{E,call}$ and $V_n^{E,put} \leq H_n^{put}$ and therefore

$$H_n^{call} - H_n^{put} \leq V_n^{E,call} - V_n^{E,put} = S_n - K\frac{1}{(1+r)^{N-n}}$$

The following inequality also holds,

$$S_n - K \leq H_n^{call} - H_n^{put}, \quad n = 0, ..., N$$

We will show this inequality by induction. For $n = N$ we have

$$H_N^{call} - X_N^{put} = S_N - K$$

Suppose that we have

$$H_n^{call} - H_n^{put} \geq S_n - K$$

and we will show that

$$H_{n-1}^{call} - H_{n-1}^{put} \geq S_{n-1} - K$$

Recall that

$$
\begin{aligned}
H_{n-1}^{put} &= \max\{(K - S_{n-1})^+, \frac{1}{1+r}\left(qH_n^{u,put} + (1-q)H_n^{d,put}\right)\} \\
H_{n-1}^{call} &= \max\{(S_{n-1} - K)^+, \frac{1}{1+r}\left(qH_n^{u,call} + (1-q)H_n^{d,call}\right)\}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&H_{n-1}^{call} - H_{n-1}^{put} \\
=\ & H_{n-1}^{call} + \min\left\{-(S_{n-1} - K)^-, \frac{-1}{1+r}\left(qH_n^{u,put} + (1-q)H_n^{d,put}\right)\right\} \\
=\ & \min\left\{H_{n-1}^{call} - (S_{n-1} - K)^-, H_{n-1}^{call} - \frac{1}{1+r}\left(qH_n^{u,put} + (1-q)H_n^{d,put}\right)\right\} \\
\geq\ & \min\left\{(S_{n-1} - K), \frac{\left(q(H_n^{u,call} - H_n^{u,put}) + (1-q)(H_n^{d,call} - H_n^{d,put})\right)}{1+r}\right\} \\
\geq\ & \min\left\{(S_{n-1} - K), \frac{(q(uS_{n-1} - K) + (1-q)(dS_{n-1} - K))}{1+r}\right\} \\
\geq\ & S_{n-1} - K
\end{aligned}
$$

We have used the obvious inequalities

$$H_{n-1}^{call} \geq (S_{n-1} - K)^+ \text{ and } H_{n-1}^{call} \geq \frac{1}{1+r}\left(qH_n^{u,call} + (1-q)H_n^{d,call}\right)$$

To sum up we have proved the following inequalities,

$$S_n - K \leq H_n^{call} - H_n^{put} \leq S_n - K\frac{1}{(1+r)^{N-n}}, \quad n = 0, ..., N$$

6.4 Bounds for options

We will show that

$$V_n^{E,call} = H_n^{call} \leq S_n$$

For $n = N$ we have

$$V_N^{E,call} = (S_N - K)^+ \leq S_N$$

We suppose that

$$V_n^{E,call} \leq S_n$$

and we will show that

$$V_{n-1}^{E,call} \leq S_{n-1}$$

We have that

$$
\begin{aligned}
V_{n-1}^{E,call} &= \frac{1}{1+r}\left(qV_n^{u,E,call} + (1-q)V_n^{d,E,call}\right) \\
&\leq \frac{1}{1+r}\left(quS_{n-1} + (1-q)dS_{n-1}\right) \\
&= S_{n-1}
\end{aligned}
$$

Next we will show that

$$V_n^{E,call} \geq S_n - K\frac{1}{(1+r)^{N-n}}$$

For $n = N$ we have that

$$V_N^{E,call} = (S_N - K)^+ \geq S_N - K$$

Suppose that we have

$$V_n^{E,call} \geq S_n - K\frac{1}{(1+r)^{N-n}}$$

We will show that

$$V_{n-1}^{E,call} \geq S_{n-1} - K\frac{1}{(1+r)^{N-n+1}}$$

We have that

$$
\begin{aligned}
V_{n-1}^{E,call} &= \frac{1}{1+r}\left(qV_n^{u,E,call} + (1-q)V_n^{d,E,call}\right) \\
&\geq \frac{1}{1+r}\left(q(uS_{n-1} - K\frac{1}{(1+r)^{N-n}}) + (1-q)(dS_{n-1} - K\frac{1}{(1+r)^{N-n}})\right) \\
&= S_{n-1} - K\frac{1}{(1+r)^{N-n+1}}
\end{aligned}
$$

Therefore we have proved so far that

$$S_n - K\frac{1}{(1+r)^{N-n}} \leq V_n^{E,call} = H_n^{call} \leq S_n$$

Next we will show that

$$K\frac{1}{(1+r)^{N-n}} - S_n \leq V_n^{E,put} \leq K\frac{1}{(1+r)^{N-n}}$$

For $n = N$ obviously we have that

$$K - S_N \leq V_N^{E,put} \leq K$$

Suppose that it holds for some n, namely,

$$K \frac{1}{(1+r)^{N-n}} - S_n \leq V_n^{E,put} \leq K \frac{1}{(1+r)^{N-n}}$$

We will show that is also holds for $n - 1$ that is

$$K \frac{1}{(1+r)^{N-n+1}} - S_{n-1} \leq V_{n-1}^{E,put} \leq K \frac{1}{(1+r)^{N-n+1}}$$

We have that

$$
\begin{aligned}
V_{n-1}^{E,put} &= \frac{1}{1+r} \left(q V_n^{u,E,put} + (1-q) V_n^{d,E,put} \right) \\
&\leq K \frac{1}{(1+r)^{N-n+1}}
\end{aligned}
$$

while

$$
\begin{aligned}
V_{n-1}^{E,put} &= \frac{1}{1+r} \left(q V_n^{u,E,put} + (1-q) V_n^{d,E,put} \right) \\
&\geq \frac{1}{1+r} \left(q(K \frac{1}{(1+r)^{N-n}} - u S_{n-1}) + (1-q)(K \frac{1}{(1+r)^{N-n}} - d S_{n-1}) \right) \\
&= K \frac{1}{(1+r)^{N-n+1}} - S_{n-1}
\end{aligned}
$$

Finally for American puts we have obviously that

$$(K - S_n)^+ \leq H_n^{put}$$

We will also show that

$$H_n^{put} \leq K$$

For $n = N$ it is obvious. We suppose that it holds for some n and we will show that

$$H_{n-1}^{put} \leq K$$

Indeed,

$$
\begin{aligned}
H_{n-1}^{put} &= \max\{(K - S_{n-1})^+, \frac{1}{1+r} \left(q H_n^{u,put} + (1-q) H_n^{d,put} \right)\} \\
&\leq \max\{(K - S_{n-1})^+, \frac{1}{1+r} K\} \\
&\leq K
\end{aligned}
$$

All the above relations and inequalities holds also in the continuous case i.e. as $n \to \infty$, in the spirit of [21] (see also a more detailed discussion of the continuous case in [10]). One can also prove, using first year calculus, that the prices for European options converges to the solution of the famous Black-Scholes-Merton formula, see for example [15], prop. 2.50.

We have proved all the above relations for the prices that the binomial model produces, but using Arbitrage arguments one can show that the same relations hold for the true prices, see for example [8].

7 Portfolio Optimization

Suppose that we are given the amount of V_0 and we are able to construct a portfolio putting money in the bank and buying a number of assets. At time zero, our portfolio is

$$V_0 = aS_0 + b \tag{7.1}$$

Suppose that we are working in an one period binomial model and our problem is how much money we will put in the bank and how many assets we should buy in order to maximize our portfolio at time 1.

Given a number $p \in (0,1)$ we want to maximize the quantity

$$pV_1^u + (1-p)V_1^d$$

where

$$
\begin{aligned}
V_1^u &= auS_0 + b(1+r), \\
V_1^d &= adS_0 + b(1+r)
\end{aligned}
$$

Of course we want the following inequalities to hold

$$
\begin{aligned}
V_1^u &\geq 0 \\
V_1^d &\geq 0
\end{aligned}
$$

and using also (2) we arrive at the following constraint on a

$$-\frac{V_0(1+r)}{S_0(u-(1+r))} \leq a \leq \frac{V_0(1+r)}{S_0(1+r-d)}$$

Note that

$$pV_1^u + (1-p)V_1^d = aS_0\big(pu + (1-p)d - (1+r)\big) + V_0(1+r)$$

and therefore if $pu + qd > 1+r$ then we optimize our quantity if we choose $a = \frac{V_0(1+r)}{S_0(1+r-d)}$ otherwise $a = -\frac{V_0(1+r)}{S_0(u-(1+r))}$.

8 Fair Value

In section 5 we have estimated the smallest value of a European option in order the writer to have enough money in any case. Is that value a fair price? If the holder of the option can construct the opposite portfolio (i.e. $(-a, -b)$) then indeed this is the fair price because both the writer and the holder can construct such a portfolios to eliminate the risk.

What about the case where the holder can not construct the opposite portfolio? Then, only the writer has eliminate the risk while the holder can lose or earn money buying this contract. Intuitively speaking this value does not seem to be fair.

For a European option the expected profit of the holder is

$$\mathbb{E}^P(X) = pX_1^u + (1-p)X_1^d$$

where X_1^u is the profit if the asset goes up and X_1^d is the profit if the asset goes down while p is the probability the asset goes up. Therefore a fair price could be the following

$$V_0 = \frac{1}{1+r} \min\left\{pX_1^u + (1-p)X_1^d, \quad qX_1^u + (1-q)X_1^d\right\}$$

where $q = \frac{1+r-d}{u-d}$.

If for a specific case $V_0 = \frac{1}{1+r}(qX_1^u + (1-q)X_1^d)$ then the writer can construct a portfolio to eliminate the risk and if $V_0 = \frac{1}{1+r}(pX_1^u + (1-p)X_1^d)$ then the writer can put this amount of money in the bank and so at time 1 this will be equal to the holder's expected profit or can construct a portfolio as we have described in Section 7.

For American type options a fair value could be

$$\min\{V_0^q, \ V_0^p\}$$

where V_0^q is the amount of money that one needs to construct a portfolio that eliminates the risk while

$$V_0^p = \max_n \frac{1}{(1+r)^n}\mathbb{E}^P(X_n)$$

where X_n is holder's profit at time n.

9 Conclusion

We described the European and American type options in discrete time using basic calculus. We explain the notion of the Arbitrage and we have seen that the fair price (when the writer and the holder can construct opposite portfolios) of an option is in fact the smallest price that the constructed portfolio should have as initial value in order the writer eliminate the risk and this smallest price is closely connected with the no Arbitrage criterion which is $d < 1 + r < u$. Furthermore we have proposed a criterion that the holder can have in mind to decide if he will exercise the option at some specific time. We show the put-call parity formulas for both European and American options and also that the values of American and European call options coincides when $r \geq 0$ while the put options coincides when $r = 0$. We also give bounds for European and American call and put options. We have discussed the portfolio's optimization problem and finally we discussed the notion of the fair value of an option when the holder can not construct an opposite portfolio to eliminate the risk.

Competing Interests

Author has declared that no competing interests exist.

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