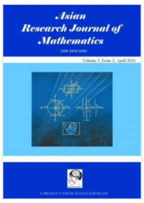

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Convolution Sums Related to Fibonacci Numbers and Lucas Numbers

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

For all natural number n we deduce binomial convolution sums formulae composed with Fibonacci numbers and Lucas numbers. Moreover we obtain those of similar convolution sums without a binomial symbol.

Keywords: Fibonacci and lucas numbers; convolution sums.

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1 Introduction

Let $\mathbb{N}$ be the set of positive integers. We may define the Fibonacci numbers, F_n, by

$$F_0 = 0, \qquad F_1 = 1, \qquad \text{and,} \qquad F_{n+2} = F_{n+1} + F_n. \tag{1.1}$$

Associated with the numbers of Fibonacci are the numbers of Lucas, L_n, which we may define by

$$L_0 = 2, \qquad L_1 = 1, \qquad \text{and,} \qquad L_{n+2} = L_{n+1} + L_n. \tag{1.2}$$

Fibonacci numbers and Lucas numbers can also be extended to negative index n satisfying

$$F_{-n} = (-1)^{n+1} F_n \qquad \text{and} \qquad L_{-n} = (-1)^n L_n.$$

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Let us observe that we may set the Fibonacci and Lucas numbers [1] by

$$F_n = \frac{a^n - b^n}{a-b}, \qquad L_n = a^n + b^n, \tag{1.3}$$

where

$$a = \frac{1}{2}(1+\sqrt{5}), \qquad b = \frac{1}{2}(1-\sqrt{5}). \tag{1.4}$$

The very general functions studied by Lucas and generalized by Bell [2], [3], are essentially the F_n and L_n defined by (1.3) with a, b being the roots of the quadratic equation $x^2 = Px - Q$ so that $a+b = P$ and $ab = Q$. In view of this formulation it is easy to show that we also have the generating function

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} F_n = \frac{e^{ax} - e^{bx}}{a-b} \tag{1.5}$$

and

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} L_n = e^{ax} + e^{bx}. \tag{1.6}$$

The above expansions enables us to deduce very easily some binomial convolution sums formulae with Fibonacci numbers and Lucas numbers. On the contrary, it is convenient for us to use (1.3) for getting other kind of convolution sums. Now Eq. (1.4) implies that

$$a + b = \frac{1}{2}(1+\sqrt{5}) + \frac{1}{2}(1-\sqrt{5}) = 1, \tag{1.7}$$

$$a - b = \frac{1}{2}(1+\sqrt{5}) - \frac{1}{2}(1-\sqrt{5}) = \sqrt{5}, \tag{1.8}$$

$$ab = \frac{1}{2}(1+\sqrt{5}) \cdot \frac{1}{2}(1-\sqrt{5}) = -1, \tag{1.9}$$

$$a^2 = 1 + a, \qquad \text{and} \qquad b^2 = 1 + b. \tag{1.10}$$

The binomial sums for Fibonacci numbers and Lucas numbers are studied by many mathematicians, for example, we can find [4], [5], and [6].

In this paper we obtain the following type of convolution sums:

$$\sum_{m=0}^{n} \binom{n}{m} m L_m L_{n-m}, \qquad \sum_{m=0}^{n} \binom{n}{m} m L_m F_{n-m},$$

and

Theorem 1.1. *Let $n \in \mathbb{N}$. Then*

(a)

$$\sum_{m=0}^{n} \binom{n}{m} m^2 L_m L_{n-m} = 2^{n-2} n(n+1) L_n + n(3n-2),$$

(b)

$$\sum_{m=0}^{n} \binom{n}{m} m^2 L_m F_{n-m} = 2^{n-2} n(n+1) F_n - n^2,$$

(c)

$$\sum_{m=0}^{n} \binom{n}{m} m^2 F_m L_{n-m} = 2^{n-2} n(n+1) F_n + n^2,$$

(d)

$$\sum_{m=0}^{n}\binom{n}{m}m^2F_mF_{n-m}=\frac{2^{n-2}n(n+1)}{5}L_n-\frac{n(3n-2)}{5}.$$

Moreover, we are interested in

$$\sum_{m=0}^{n} mF_mL_{n-m}, \qquad \sum_{m=0}^{n} mF_mF_{n-m},$$

etc., and

Theorem 1.2. *Let $n\in\mathbb{N}$. Then*

(a)

$$\sum_{m=0}^{n} m^2L_mL_{n-m}=\frac{n(n+1)(2n+1)}{6}L_n+\frac{n(5n+4)}{5}F_{n+1} - \frac{2(n+1)}{5}F_n,$$

(b)

$$\sum_{m=0}^{n} m^2L_mF_{n-m}=-\frac{n^2}{5}F_{n+1}+\frac{n(n+1)(10n-7)}{30}F_n,$$

(c)

$$\sum_{m=0}^{n} m^2F_mL_{n-m}=\frac{n^2}{5}F_{n+1}+\frac{n(n+1)(10n+17)}{30}F_n,$$

(d)

$$\sum_{m=0}^{n} m^2F_mF_{n-m}=\frac{n(n+1)(2n+1)}{30}L_n-\frac{n(5n+4)}{25}F_{n+1} + \frac{2(n+1)}{25}F_n.$$

2 Binomial Convolution Sums

Lemma 2.1. *Let $n\in\mathbb{N}$. Then*

(a)

$$\frac{ae^{2ax}-be^{2bx}}{a-b}=\sum_{n=0}^{\infty}2^nF_{n+1}\frac{x^n}{n!},$$

(b)

$$ae^{2ax}+be^{2bx}=\sum_{n=0}^{\infty}2^nL_{n+1}\frac{x^n}{n!}.$$

Proof. Since the proofs are similar so we prove only part (a). From (1.5) we note that

$$\begin{aligned}
\frac{ae^{2ax}-be^{2bx}}{a-b} &= \frac{1}{2}\cdot\frac{2ae^{2ax}-2be^{2bx}}{a-b} \\
&= \frac{1}{2}\cdot\frac{d}{dx}\left(\frac{e^{2ax}-e^{2bx}}{a-b}\right) \\
&= \frac{1}{2}\cdot\frac{1}{a-b}\cdot\frac{d}{dx}\left(\sum_{k=0}^{\infty}\frac{x^k}{k!}\left(2^ka^k-2^kb^k\right)\right) \\
&= \frac{1}{2}\cdot\sum_{k=1}^{\infty}k\frac{x^{k-1}}{k!}\cdot 2^k\cdot\frac{a^k-b^k}{a-b} \\
&= \frac{1}{2}\cdot\sum_{k=1}^{\infty}\frac{x^{k-1}}{k!}\cdot 2^kkF_k \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\frac{x^n}{(n+1)!}\cdot 2^{n+1}(n+1)F_{n+1} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\frac{x^n}{n!}\cdot 2^{n+1}F_{n+1} \\
&= \sum_{n=0}^{\infty}2^nF_{n+1}\frac{x^n}{n!}.
\end{aligned}$$

□

The following proposition is also already obtained in the literature [4] and [7].

Proposition 2.1. *(See [8]) Let $n \in \mathbb{N}$. Then*

(a)

$$\sum_{m=0}^{n}\binom{n}{m}L_mL_{n-m} = 2^nL_n + 2,$$

(b)

$$\sum_{m=0}^{n}\binom{n}{m}L_mF_{n-m} = 2^nF_n,$$

(c)

$$\sum_{m=0}^{n}\binom{n}{m}F_mF_{n-m} = \frac{2^n}{5}L_n - \frac{2}{5}.$$

Theorem 2.2. *Let $n \in \mathbb{N}$. Then*

(a)

$$\sum_{m=0}^{n}\binom{n}{m}mL_mL_{n-m} = 2^{n-1}nL_n + n,$$

(b)

$$\sum_{m=0}^{n}\binom{n}{m}mL_mF_{n-m} = 2^{n-1}nF_n - n,$$

(c)

$$\sum_{m=0}^{n}\binom{n}{m}mF_mL_{n-m}=2^{n-1}nF_n+n,$$

(d)

$$\sum_{m=0}^{n}\binom{n}{m}mF_mF_{n-m}=\frac{2^{n-1}nL_n}{5}-\frac{n}{5}.$$

Proof. (a) Now we observe that

$$\sum_{m=0}^{n}\binom{n}{m}mL_mL_{n-m}=\sum_{m=0}^{n}\binom{n}{m}(n-m)L_{n-m}L_m$$

and so by Proposition 2.1 (a) we have

$$\begin{aligned}\sum_{m=0}^{n}\binom{n}{m}mL_mL_{n-m}&=\frac{n}{2}\sum_{m=0}^{n}\binom{n}{m}L_mL_{n-m}\\&=\frac{n}{2}\left(2^nL_n+2\right)\\&=2^{n-1}nL_n+n.\end{aligned}$$

(b) By (1.5), (1.6), and (1.7) we obtain

$$\begin{aligned}&\sum_{N=0}^{\infty}\left(\sum_{m=0}^{N+1}\binom{N+1}{m}mL_mF_{N+1-m}\right)\cdot\frac{1}{N+1}\cdot\frac{x^N}{N!}\\&=\sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\binom{n}{m}mL_mF_{n-m}\right)\cdot\frac{x^{n-1}}{n!}\\&=\left(\sum_{k=0}^{\infty}\frac{x^k}{k!}F_k\right)\left(\sum_{m=1}^{\infty}\frac{x^{m-1}}{m!}mL_m\right)\\&=\frac{e^{ax}-e^{bx}}{a-b}\cdot\frac{d}{dx}\left(e^{ax}+e^{bx}\right)\\&=\frac{e^{ax}-e^{bx}}{a-b}\cdot\left(ae^{ax}+be^{bx}\right)\\&=\frac{ae^{2ax}+be^{(a+b)x}-ae^{(b+a)x}-be^{2bx}}{a-b}\\&=\frac{ae^{2ax}-be^{2bx}}{a-b}-e^x\end{aligned}$$

then applying Lemma 2.1 (a) to the above identity we have

$$\begin{aligned}&\sum_{N=0}^{\infty}\left(\sum_{m=0}^{N+1}\binom{N+1}{m}mL_mF_{N+1-m}\right)\cdot\frac{1}{N+1}\cdot\frac{x^N}{N!}\\&=\sum_{N=0}^{\infty}2^NF_{N+1}\frac{x^N}{N!}-\sum_{N=0}^{\infty}\frac{x^N}{N!}\\&=\sum_{N=0}^{\infty}\frac{x^N}{N!}\left(2^NF_{N+1}-1\right)\end{aligned}$$

and

$$\sum_{m=0}^{N+1}\binom{N+1}{m}mL_mF_{N+1-m}\cdot\frac{1}{N+1}=2^NF_{N+1}-1.$$

By letting $n=N+1$ we conclude that

$$\sum_{m=0}^{n}\binom{n}{m}mL_mF_{n-m}=n\left(2^{n-1}F_n-1\right).$$

(c) From Proposition 2.1 (b) and Theorem 2.2 (b) we expand as follows :

$$\begin{aligned}\sum_{m=0}^{n}\binom{n}{m}mF_mL_{n-m}&=\sum_{m=0}^{n}\binom{n}{m}(n-m)F_{n-m}L_m\\&=n\sum_{m=0}^{n}\binom{n}{m}L_mF_{n-m}-\sum_{m=0}^{n}\binom{n}{m}mL_mF_{n-m}\\&=n\cdot 2^nF_n-\left(2^{n-1}nF_n-n\right)\\&=2^{n-1}nF_n+n.\end{aligned}$$

(d) It is similar to part (a) except for using Proposition 2.1 (c).

□

Proof of Theorem 1.1. (a) In a similar manner, by (1.6), (1.7), and (1.10) we can easily know that

$$\begin{aligned}&\sum_{N=0}^{\infty}\left(\sum_{m=0}^{N+2}\binom{N+2}{m}m(m-1)L_mL_{N+2-m}\right)\cdot\frac{1}{(N+2)(N+1)}\cdot\frac{x^N}{N!}\\&=\sum_{n=2}^{\infty}\left(\sum_{m=2}^{n}\binom{n}{m}m(m-1)L_mL_{n-m}\right)\cdot\frac{x^{n-2}}{n!}\\&=\left(\sum_{k=0}^{\infty}\frac{x^k}{k!}L_k\right)\left(\sum_{m=2}^{\infty}\frac{x^{m-2}}{m!}m(m-1)L_m\right)\\&=\left(e^{ax}+e^{bx}\right)\cdot\frac{d^2}{dx^2}\left(e^{ax}+e^{bx}\right)\\&=\left(e^{ax}+e^{bx}\right)\left(a^2e^{ax}+b^2e^{bx}\right)\\&=a^2e^{2ax}+b^2e^{(a+b)x}+a^2e^{(b+a)x}+b^2e^{2bx}\\&=(1+a)e^{2ax}+(1+b)e^{x}+(1+a)e^{x}+(1+b)e^{2bx}\\&=\left(e^{2ax}+e^{2bx}\right)+\left(ae^{2ax}+be^{2bx}\right)+3e^{x},\end{aligned}$$

which requires Lemma 2.1 (b) thus we have

$$\begin{aligned}&\sum_{N=0}^{\infty}\left(\sum_{m=0}^{N+2}\binom{N+2}{m}m(m-1)L_mL_{N+2-m}\right)\cdot\frac{1}{(N+2)(N+1)}\cdot\frac{x^N}{N!}\\&=\sum_{N=0}^{\infty}\frac{x^N}{N!}\left(2^NL_N+2^NL_{N+1}+3\right)\end{aligned}$$

Here replacing $N+2$ with n we obtain

$$\sum_{m=0}^{n}\binom{n}{m}m(m-1)L_mL_{n-m}=n(n-1)\left(2^{n-2}L_{n-2}+2^{n-2}L_{n-1}+3\right).$$

Appealing to Theorem 2.2 (a) and

$$L_n=L_{n-1}+L_{n-2},$$

we show that

$$\begin{aligned}
&\sum_{m=0}^{n}\binom{n}{m}m^2L_mL_{n-m}\\
&=n(n-1)\left(2^{n-2}L_{n-2}+2^{n-2}L_{n-1}+3\right)+\sum_{m=0}^{n}\binom{n}{m}mL_mL_{n-m}\\
&=n(n-1)\left(2^{n-2}L_{n-2}+2^{n-2}L_{n-1}+3\right)+2^{n-1}nL_n+n\\
&=2^{n-2}n(n-1)L_{n-2}+2^{n-2}n(n-1)L_{n-1}+3n(n-1)\\
&\quad+2^{n-1}nL_n+n\\
&=2^{n-2}n(n-1)\left(L_n-L_{n-1}\right)+2^{n-2}n(n-1)L_{n-1}+n(3n-2)\\
&\quad+2^{n-1}nL_n\\
&=2^{n-2}n(n+1)L_n+n(3n-2).
\end{aligned}$$

(b) From (1.5), (1.6), (1.7), and (1.10) we note that

$$\begin{aligned}
&\sum_{N=0}^{\infty}\left(\sum_{m=0}^{N+2}\binom{N+2}{m}m(m-1)L_mF_{N+2-m}\right)\cdot\frac{1}{(N+2)(N+1)}\cdot\frac{x^N}{N!}\\
&=\sum_{n=2}^{\infty}\left(\sum_{m=2}^{n}\binom{n}{m}m(m-1)L_mF_{n-m}\right)\cdot\frac{x^{n-2}}{n!}\\
&=\left(\sum_{k=0}^{\infty}\frac{x^k}{k!}F_k\right)\left(\sum_{m=2}^{\infty}\frac{x^{m-2}}{m!}m(m-1)L_m\right)\\
&=\frac{e^{ax}-e^{bx}}{a-b}\cdot\frac{d^2}{dx^2}\left(e^{ax}+e^{bx}\right)\\
&=\frac{e^{ax}-e^{bx}}{a-b}\cdot\left(a^2e^{ax}+b^2e^{bx}\right)\\
&=\frac{a^2e^{2ax}+b^2e^{(a+b)x}-a^2e^{(b+a)x}-b^2e^{2bx}}{a-b}\\
&=\frac{(1+a)e^{2ax}-(1+b)e^{2bx}}{a-b}-\frac{a^2-b^2}{a-b}e^x\\
&=\frac{e^{2ax}-e^{2bx}}{a-b}+\frac{ae^{2ax}-be^{2bx}}{a-b}-e^x
\end{aligned}$$

then using Lemma 2.1 (a) we deduce that

$$\sum_{N=0}^{\infty}\left(\sum_{m=0}^{N+2}\binom{N+2}{m}m(m-1)L_mF_{N+2-m}\right)\cdot\frac{1}{(N+2)(N+1)}\cdot\frac{x^N}{N!}$$
$$=\sum_{N=0}^{\infty}2^NF_N\frac{x^N}{N!}+\sum_{N=0}^{\infty}2^NF_{N+1}\frac{x^N}{N!}-\sum_{N=0}^{\infty}\frac{x^N}{N!}$$
$$=\sum_{N=0}^{\infty}\frac{x^N}{N!}\left(2^NF_N+2^NF_{N+1}-1\right)$$

and

$$\sum_{m=0}^{N+2}\binom{N+2}{m}m(m-1)L_mF_{N+2-m}\cdot\frac{1}{(N+2)(N+1)}$$
$$=2^NF_N+2^NF_{N+1}-1.$$

Therefore replacing $N+2$ with n we get

$$\sum_{m=0}^{n}\binom{n}{m}m(m-1)L_mF_{n-m}=n(n-1)\left(2^{n-2}F_{n-2}+2^{n-2}F_{n-1}-1\right)$$

and so applying Theorem 2.2 (b) we have

$$\sum_{m=0}^{n}\binom{n}{m}m^2L_mF_{n-m}$$
$$=n(n-1)\left(2^{n-2}F_{n-2}+2^{n-2}F_{n-1}-1\right)+\sum_{m=0}^{n}\binom{n}{m}mL_mF_{n-m}$$
$$=n(n-1)\left(2^{n-2}F_{n-2}+2^{n-2}F_{n-1}-1\right)+n\left(2^{n-1}F_n-1\right)$$
$$=2^{n-2}n(n-1)F_{n-2}+2^{n-2}n(n-1)F_{n-1}+2^{n-1}nF_n-n^2$$
$$=2^{n-2}n(n-1)\left(F_n-F_{n-1}\right)+2^{n-2}n(n-1)F_{n-1}$$
$$+2^{n-1}nF_n-n^2$$
$$=2^{n-2}n(n+1)F_n-n^2.$$

For the last line in the above identity we refer to

$$F_n=F_{n-1}+F_{n-2}.$$

(c) By Proposition 2.1 (b), Theorem 2.2 (b), and Theorem 1.1 (b) we consider:

$$\sum_{m=0}^{n}\binom{n}{m}m^2F_mL_{n-m}$$
$$=\sum_{m=0}^{n}\binom{n}{m}(n-m)^2F_{n-m}L_m$$

$$
\begin{aligned}
&= \sum_{m=0}^{n} \binom{n}{m} \left(n^2 - 2mn + m^2\right) L_m F_{n-m} \\
&= n^2 \sum_{m=0}^{n} \binom{n}{m} L_m F_{n-m} - 2n \sum_{m=0}^{n} \binom{n}{m} m L_m F_{n-m} \\
&\quad + \sum_{m=0}^{n} \binom{n}{m} m^2 L_m F_{n-m} \\
&= n^2 \cdot 2^n F_n - 2n \left(2^{n-1} n F_n - n\right) + 2^{n-2} n(n+1) F_n - n^2 \\
&= 2^{n-2} n(n+1) F_n + n^2.
\end{aligned}
$$

(d) It is similar to part (a) except for using Theorem 2.2 (d).

□

3 Convolution sums with Fibonacci Numbers and Lucas Numbers

Proposition 3.1. *(See [8]) Let $n \in \mathbb{N}$. Then*

(a)

$$\sum_{m=0}^{n} L_m L_{n-m} = (n+1) L_n + 2F_{n+1},$$

(b)

$$\sum_{m=0}^{n} L_m F_{n-m} = (n+1) F_n,$$

(c)

$$\sum_{m=0}^{n} F_m F_{n-m} = \frac{1}{5}(n+1) L_n - \frac{2}{5} F_{n+1}.$$

Theorem 3.1. *Let $n \in \mathbb{N}$. Then*

(a)

$$\sum_{m=0}^{n} m L_m L_{n-m} = \frac{n(n+1)}{2} L_n + n F_{n+1},$$

(b)

$$\sum_{m=0}^{n} m L_m F_{n-m} = -\frac{1}{5} n F_{n+1} + \frac{(5n-4)(n+1)}{10} F_n,$$

(c)

$$\sum_{m=0}^{n} m F_m L_{n-m} = \frac{1}{5} n F_{n+1} + \frac{(5n+4)(n+1)}{10} F_n,$$

(d)

$$\sum_{m=0}^{n} mF_mF_{n-m} = \frac{n(n+1)}{10}L_n - \frac{n}{5}F_{n+1}.$$

Proof. (a) Now we easily note that

$$\sum_{m=0}^{n} mL_mL_{n-m} = \sum_{m=0}^{n}(n-m)L_{n-m}L_m$$

and so by Proposition 3.1 (a) we have

$$\begin{aligned}\sum_{m=0}^{n} mL_mL_{n-m} &= \frac{n}{2}\sum_{m=0}^{n} L_mL_{n-m} \\ &= \frac{n}{2}\Big((n+1)L_n + 2F_{n+1}\Big) \\ &= \frac{n(n+1)}{2}L_n + nF_{n+1}.\end{aligned}$$

(b) By (1.3) we deduce the following convolution sum:

$$\begin{aligned}&\sum_{m=0}^{n} mL_mF_{n-m} \\ &= \sum_{m=0}^{n} m\left(a^m+b^m\right)\cdot\frac{a^{n-m}-b^{n-m}}{a-b} \\ &= \sum_{m=0}^{n} m\cdot\frac{a^n-a^mb^{n-m}+b^ma^{n-m}-b^n}{a-b} \\ &= \frac{a^n-b^n}{a-b}\sum_{m=0}^{n} m - \frac{b^n}{a-b}\sum_{m=0}^{n} m(\frac{a}{b})^m + \frac{a^n}{a-b}\sum_{m=0}^{n} m(\frac{b}{a})^m.\end{aligned} \tag{3.1}$$

Then by the geometric series we can calculate the second term in the above identity as

$$\begin{aligned}&\frac{b^n}{a-b}\sum_{m=0}^{n} m(\frac{a}{b})^m \\ &= \frac{b^n}{a-b}\left(\frac{a}{b} + 2(\frac{a}{b})^2 + 3(\frac{a}{b})^3 + \cdots + n(\frac{a}{b})^n\right) \\ &= \frac{b^n}{a-b}\Bigg[\Big\{\frac{a}{b} + (\frac{a}{b})^2 + (\frac{a}{b})^3 + \cdots + (\frac{a}{b})^n\Big\} + \Big\{(\frac{a}{b})^2 + (\frac{a}{b})^3 + \cdots + (\frac{a}{b})^n\Big\} \\ &\quad + \Big\{(\frac{a}{b})^3 + \cdots + (\frac{a}{b})^3\Big\} + \cdots + (\frac{a}{b})^n\Bigg] \\ &= \frac{b^n}{a-b}\Bigg\{\frac{\frac{a}{b}\Big((\frac{a}{b})^n - 1\Big)}{\frac{a}{b}-1} + \frac{(\frac{a}{b})^2\Big((\frac{a}{b})^{n-1} - 1\Big)}{\frac{a}{b}-1} + \frac{(\frac{a}{b})^3\Big((\frac{a}{b})^{n-2} - 1\Big)}{\frac{a}{b}-1} \\ &\quad + \cdots + (\frac{a}{b})^n\Bigg\}\end{aligned}$$

$$= \frac{b^n}{a-b}\left(\frac{ab}{b^{n+1}} \cdot \frac{a^n - b^n}{a-b} + \frac{ab \cdot a}{b^{n+1}} \cdot \frac{a^{n-1} - b^{n-1}}{a-b} + \frac{ab \cdot a^2}{b^{n+1}} \cdot \frac{a^{n-2} - b^{n-2}}{a-b}\right.$$
$$\left. + \cdots + \frac{ab \cdot a^{n-1}}{b^{n+1}} \cdot 1\right)$$
$$= \frac{1}{(a-b)b}\Big(-a^0 F_n - aF_{n-1} - a^2 F_{n-2} - \cdots - a^{n-1} F_1\Big)$$
$$= -\frac{1}{(a-b)b} \sum_{i=0}^{n-1} a^i F_{n-i}$$

similarly the third term becomes

$$\frac{a^n}{a-b} \sum_{m=0}^{n} m\left(\frac{b}{a}\right)^m = -\frac{1}{(a-b)a} \sum_{i=0}^{n-1} b^i F_{n-i}.$$

Therefore applying the above results to Eq. (3.1) we obtain

$$\begin{aligned} &\sum_{m=0}^{n} mL_m F_{n-m} \\ &= \frac{a^n - b^n}{a-b} \sum_{m=0}^{n} m + \frac{1}{(a-b)b} \sum_{i=0}^{n-1} a^i F_{n-i} - \frac{1}{(a-b)a} \sum_{i=0}^{n-1} b^i F_{n-i} \\ &= F_n \cdot \frac{n(n+1)}{2} - \frac{\sum_{i=0}^{n-1} a^i F_{n-i}}{2+b} - \frac{\sum_{i=0}^{n-1} b^i F_{n-i}}{2+a}, \end{aligned} \tag{3.2}$$

where by (1.9) and (1.10) we use

$$(a-b)b = ab - b^2 = -1 - (1+b) = -2 - b \tag{3.3}$$

and

$$(a-b)a = a^2 - ba = (1+a) - (-1) = 2 + a. \tag{3.4}$$

Now by (1.1), (1.2), and Proposition 3.1 (b) the second and third term of (3.2) are equal to

$$\begin{aligned} &- \frac{\sum_{i=0}^{n-1} a^i F_{n-i}}{2+b} - \frac{\sum_{i=0}^{n-1} b^i F_{n-i}}{2+a} \\ &= -\frac{(2+a)\sum_{i=0}^{n-1} a^i F_{n-i} + (2+b)\sum_{i=0}^{n-1} b^i F_{n-i}}{(2+b)(2+a)} \\ &= -\frac{2\sum_{i=0}^{n-1}\left(a^i + b^i\right) F_{n-i} + \sum_{i=0}^{n-1}\left(a^{i+1} + b^{i+1}\right) F_{n-i}}{4 + 2(a+b) + ab} \end{aligned}$$

$$= -\frac{2\sum_{i=0}^{n-1} L_i F_{n-i} + \sum_{i=0}^{n-1} L_{i+1} F_{n-i}}{5}$$
$$= -\frac{2\left(\sum_{i=0}^{n} L_i F_{n-i} - L_n F_0\right)}{5}$$
$$-\frac{\sum_{m=0}^{n+1} L_m F_{n+1-m} - L_0 F_{n+1} - L_{n+1} F_0}{5}$$
$$= -\frac{2(n+1)F_n}{5} - \frac{(n+2)F_{n+1} - 2F_{n+1}}{5}$$
$$= -\frac{2(n+1)F_n}{5} - \frac{nF_{n+1}}{5}.$$

Thus Eq. (3.2) is

$$\sum_{m=0}^{n} mL_m F_{n-m} = \frac{n(n+1)}{2}F_n - \frac{2(n+1)F_n}{5} - \frac{nF_{n+1}}{5}$$
$$= -\frac{1}{5}nF_{n+1} + \frac{(5n-4)(n+1)}{10}F_n.$$

(c) From Proposition 3.1 (b) and Theorem 3.1 (b) we can consider

$$\sum_{m=0}^{n} mF_m L_{n-m} = \sum_{m=0}^{n} (n-m)\, F_{n-m} L_m$$
$$= n\sum_{m=0}^{n} L_m F_{n-m} - \sum_{m=0}^{n} mL_m F_{n-m}$$
$$= n\cdot(n+1)F_n - \left(-\frac{1}{5}nF_{n+1} + \frac{(5n-4)(n+1)}{10}F_n\right)$$
$$= \frac{1}{5}nF_{n+1} + \frac{(5n+4)(n+1)}{10}F_n.$$

(d) It is similar to part (a) except for using Proposition 3.1 (c).

□

Proof of Theorem 1.2. In advance, by (1.9) let us investigate as follows:

$$\sum_{m=0}^{n} m^2 (\frac{a}{b})^m$$
$$= \frac{a}{b} + 2^2(\frac{a}{b})^2 + 3^2(\frac{a}{b})^3 + \cdots + n^2(\frac{a}{b})^n$$
$$= \left\{\frac{a}{b} + (\frac{a}{b})^2 + (\frac{a}{b})^3 + \cdots + (\frac{a}{b})^n\right\} + 3\left\{(\frac{a}{b})^2 + (\frac{a}{b})^3 + \cdots + (\frac{a}{b})^n\right\}$$
$$+ 5\left\{(\frac{a}{b})^3 + \cdots + (\frac{a}{b})^n\right\} + \cdots + (2n-1)(\frac{a}{b})^n$$
$$= \frac{\frac{a}{b}\left((\frac{a}{b})^n - 1\right)}{\frac{a}{b} - 1} + 3\cdot\frac{(\frac{a}{b})^2\left((\frac{a}{b})^{n-1} - 1\right)}{\frac{a}{b} - 1} + 5\cdot\frac{(\frac{a}{b})^3\left((\frac{a}{b})^{n-2} - 1\right)}{\frac{a}{b} - 1}$$
$$+ \cdots + (2n-1)\cdot(\frac{a}{b})^n$$

$$
\begin{aligned}
&= \frac{ab}{b^{n+1}} \cdot \frac{a^n - b^n}{a - b} + 3 \cdot \frac{ab \cdot a}{b^{n+1}} \cdot \frac{a^{n-1} - b^{n-1}}{a - b} + 5 \cdot \frac{ab \cdot a^2}{b^{n+1}} \cdot \frac{a^{n-2} - b^{n-2}}{a - b} \\
&\quad + \cdots + (2n-1) \cdot \frac{ab \cdot a^{n-1}}{b^{n+1}} \cdot 1 \\
&= -\frac{1}{b^{n+1}} \Big(1 \cdot a^0 F_n + 3aF_{n-1} + 5a^2 F_{n-2} + \cdots + (2n-1)a^{n-1} F_1 \Big) \\
&= -\frac{1}{b^{n+1}} \sum_{i=0}^{n-1} (2i+1) a^i F_{n-i}
\end{aligned} \tag{3.5}
$$

and similarly,

$$
\sum_{m=0}^{n} m^2 (\frac{b}{a})^m = -\frac{1}{a^{n+1}} \sum_{i=0}^{n-1} (2i+1) b^i F_{n-i}. \tag{3.6}
$$

(a) By (1.3), (3.5), and (3.6) we can observe that

$$
\begin{aligned}
&\sum_{m=0}^{n} m^2 L_m L_{n-m} \\
&= \sum_{m=0}^{n} m^2 \left(a^m + b^m\right) \left(a^{n-m} + b^{n-m}\right) \\
&= \sum_{m=0}^{n} m^2 \left(a^n + a^m b^{n-m} + b^m a^{n-m} + b^n\right) \\
&= \left(a^n + b^n\right) \sum_{m=0}^{n} m^2 + b^n \sum_{m=0}^{n} m^2 (\frac{a}{b})^m + a^n \sum_{m=0}^{n} m^2 (\frac{b}{a})^m \\
&= L_n \cdot \frac{n(n+1)(2n+1)}{6} + b^n \left\{ -\frac{1}{b^{n+1}} \sum_{i=0}^{n-1} (2i+1) a^i F_{n-i} \right\} \\
&\quad + a^n \left\{ -\frac{1}{a^{n+1}} \sum_{i=0}^{n-1} (2i+1) b^i F_{n-i} \right\}
\end{aligned} \tag{3.7}
$$

Then by (1.9), the second and third term in the above equation becomes

$$
\begin{aligned}
&b^n \left\{ -\frac{1}{b^{n+1}} \sum_{i=0}^{n-1} (2i+1) a^i F_{n-i} \right\} + a^n \left\{ -\frac{1}{a^{n+1}} \sum_{i=0}^{n-1} (2i+1) b^i F_{n-i} \right\} \\
&= -\frac{1}{ab} \sum_{i=0}^{n-1} (2i+1) a^{i+1} F_{n-i} - \frac{1}{ab} \sum_{i=0}^{n-1} (2i+1) b^{i+1} F_{n-i} \\
&= \sum_{i=0}^{n-1} (2i+1) \left(a^{i+1} + b^{i+1}\right) F_{n-i} \\
&= \sum_{i=0}^{n-1} (2i+1) L_{i+1} F_{n-i} \\
&= \sum_{m=1}^{n} \left(2(m-1)+1\right) L_m F_{n+1-m}
\end{aligned}
$$

$$
\begin{aligned}
&= 2\sum_{m=1}^{n} mL_m F_{n+1-m} - \sum_{m=1}^{n} L_m F_{n+1-m} \\
&= 2\left(\sum_{m=0}^{n+1} mL_m F_{n+1-m} - (n+1)L_{n+1}F_0\right) \\
&\quad - \left(\sum_{m=0}^{n+1} L_m F_{n+1-m} - L_0 F_{n+1} - L_{n+1}F_0\right)
\end{aligned}
$$

and so appealing to (1.1), (1.2), Proposition 3.1 (b), and Theorem 3.1 (b) we have

$$
\begin{aligned}
& b^n\left\{-\frac{1}{b^{n+1}}\sum_{i=0}^{n-1}(2i+1)a^i F_{n-i}\right\} + a^n\left\{-\frac{1}{a^{n+1}}\sum_{i=0}^{n-1}(2i+1)b^i F_{n-i}\right\} \\
&= \sum_{i=0}^{n-1}(2i+1)L_{i+1}F_{n-i} \\
&= 2\left(-\frac{1}{5}(n+1)F_{n+2} + \frac{(5n+1)(n+2)}{10}F_{n+1}\right) \\
&\quad - \Big((n+2)F_{n+1} - 2F_{n+1}\Big) \\
&= -\frac{2}{5}(n+1)(F_{n+1}+F_n) + \frac{(5n+1)(n+2)}{5}F_{n+1} - nF_{n+1} \\
&= -\frac{2}{5}(n+1)F_n + \frac{n(5n+4)}{5}F_{n+1}.
\end{aligned} \tag{3.8}
$$

Therefore applying (3.8) to Eq. (3.7) completes the proof.

(b) Eq. (1.3), (3.3), (3.4), (3.5), and (3.6) implies that

$$
\begin{aligned}
& \sum_{m=0}^{n} m^2 L_m F_{n-m} \\
&= \sum_{m=0}^{n} m^2\left(a^m + b^m\right)\cdot\frac{a^{n-m} - b^{n-m}}{a-b} \\
&= \frac{1}{a-b}\sum_{m=0}^{n} m^2\left(a^n - a^m b^{n-m} + b^m a^{n-m} - b^n\right) \\
&= \frac{a^n - b^n}{a-b}\sum_{m=0}^{n} m^2 - \frac{b^n}{a-b}\sum_{m=0}^{n} m^2\left(\frac{a}{b}\right)^m + \frac{a^n}{a-b}\sum_{m=0}^{n} m^2\left(\frac{b}{a}\right)^m \\
&= F_n\cdot\frac{n(n+1)(2n+1)}{6} - \frac{b^n}{a-b}\left\{-\frac{1}{b^{n+1}}\sum_{i=0}^{n-1}(2i+1)a^i F_{n-i}\right\} \\
&\quad + \frac{a^n}{a-b}\left\{-\frac{1}{a^{n+1}}\sum_{i=0}^{n-1}(2i+1)b^i F_{n-i}\right\} \\
&= \frac{n(n+1)(2n+1)}{6}F_n + \frac{1}{ab-b^2}\sum_{i=0}^{n-1}(2i+1)a^i F_{n-i} \\
&\quad - \frac{1}{a^2-ab}\sum_{i=0}^{n-1}(2i+1)b^i F_{n-i}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{n(n+1)(2n+1)}{6}F_n - \frac{1}{2+b}\sum_{i=0}^{n-1}(2i+1)a^iF_{n-i} \\
&\quad - \frac{1}{2+a}\sum_{i=0}^{n-1}(2i+1)b^iF_{n-i} \\
&= \frac{n(n+1)(2n+1)}{6}F_n \\
&\quad - \frac{(2+a)\sum_{i=0}^{n-1}(2i+1)a^iF_{n-i} + (2+b)\sum_{i=0}^{n-1}(2i+1)b^iF_{n-i}}{(2+b)(2+a)} \\
&= \frac{n(n+1)(2n+1)}{6}F_n - \frac{2\sum_{i=0}^{n-1}(2i+1)\left(a^i+b^i\right)F_{n-i}}{5} \\
&\quad - \frac{\sum_{i=0}^{n-1}(2i+1)\left(a^{i+1}+b^{i+1}\right)F_{n-i}}{5}
\end{aligned}
$$

and so

$$
\begin{aligned}
\sum_{m=0}^{n} m^2L_mF_{n-m} = \frac{n(n+1)(2n+1)}{6}F_n - \frac{2\sum_{i=0}^{n-1}(2i+1)L_iF_{n-i}}{5} \\
- \frac{\sum_{i=0}^{n-1}(2i+1)L_{i+1}F_{n-i}}{5}.
\end{aligned}
$$

Then we refer to (3.8) and for the last step we use (1.1), Proposition 3.1 (b), and Theorem 3.1 (b) to apply as follows:

$$
\begin{aligned}
\sum_{i=0}^{n-1}(2i+1)L_iF_{n-i} &= \sum_{i=0}^{n}(2i+1)L_iF_{n-i} - (2n+1)L_nF_0 \\
&= \sum_{i=0}^{n}(2i+1)L_iF_{n-i} \\
&= 2\sum_{i=0}^{n} iL_iF_{n-i} + \sum_{i=0}^{n} L_iF_{n-i} \\
&= 2\left(-\frac{1}{5}nF_{n+1} + \frac{(5n-4)(n+1)}{10}F_n\right) \\
&\quad + (n+1)F_n \\
&= -\frac{2}{5}nF_{n+1} + \frac{(5n+1)(n+1)}{5}F_n.
\end{aligned}
$$

(c) Now we can expand the desired convolution sum as

$$\sum_{m=0}^{n} m^2 F_m L_{n-m} = \sum_{m=0}^{n} (n-m)^2 F_{n-m} L_m$$
$$= n^2 \sum_{m=0}^{n} L_m F_{n-m} - 2n \sum_{m=0}^{n} m L_m F_{n-m}$$
$$+ \sum_{m=0}^{n} m^2 L_m F_{n-m}.$$

Finally we use Proposition 3.1 (b), Theorem 3.1 (b), and Theorem 1.2 (b).

(d) From (1.3), (1.8), (1.9), (3.5), and (3.6) we easily know that

$$\sum_{m=0}^{n} m^2 F_m F_{n-m}$$
$$= \sum_{m=0}^{n} m^2 \frac{a^m - b^m}{a-b} \cdot \frac{a^{n-m} - b^{n-m}}{a-b}$$
$$= \frac{1}{(a-b)^2} \sum_{m=0}^{n} m^2 \left(a^n - a^m b^{n-m} - b^m a^{n-m} + b^n\right)$$
$$= \frac{a^n + b^n}{5} \sum_{m=0}^{n} m^2 - \frac{b^n}{5} \sum_{m=0}^{n} m^2 (\frac{a}{b})^m - \frac{a^n}{5} \sum_{m=0}^{n} m^2 (\frac{b}{a})^m$$
$$= \frac{L_n}{5} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{b^n}{5} \left\{ -\frac{1}{b^{n+1}} \sum_{i=0}^{n-1} (2i+1) a^i F_{n-i} \right\}$$
$$- \frac{a^n}{5} \left\{ -\frac{1}{a^{n+1}} \sum_{i=0}^{n-1} (2i+1) b^i F_{n-i} \right\}$$
$$= \frac{n(n+1)(2n+1)}{30} L_n + \frac{1}{5b} \sum_{i=0}^{n-1} (2i+1) a^i F_{n-i} + \frac{1}{5a} \sum_{i=0}^{n-1} (2i+1) b^i F_{n-i}$$
$$= \frac{n(n+1)(2n+1)}{30} L_n - \frac{1}{5} \sum_{i=0}^{n-1} (2i+1) a^{i+1} F_{n-i}$$
$$- \frac{1}{5} \sum_{i=0}^{n-1} (2i+1) b^{i+1} F_{n-i}$$
$$= \frac{n(n+1)(2n+1)}{30} L_n - \frac{1}{5} \sum_{i=0}^{n-1} (2i+1) \left(a^{i+1} + b^{i+1}\right) F_{n-i}$$
$$= \frac{n(n+1)(2n+1)}{30} L_n - \frac{1}{5} \sum_{i=0}^{n-1} (2i+1) L_{i+1} F_{n-i}$$

so we refer to (3.8).

□

4 Conclusion

Here we newly make the convolution sums as

$$\sum_{m=0}^{n} m^2 L_m L_{n-m}, \qquad \sum_{m=0}^{n} m^2 L_m F_{n-m}, \qquad \sum_{m=0}^{n} m^2 F_m L_{n-m},$$

and

$$\sum_{m=0}^{n} m^2 F_m F_{n-m}$$

so that we obtain their formulae.

Competing Interests

Author has declared that no competing interests exist.

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