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Common Fixed Point Theorems for Compatible and Weakly Compatible Maps Satisfying E. A. and CLRT Property in Non-Newtonian Metric Space

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

In this paper, we prove some results for compatible and weakly compatible maps in non-Newtonian metric spaces. We also introduce E.A. and CLRT property in the context of non-Newtonian metric space and prove the corresponding fixed point results. We also provide examples to illustrate the concepts.

Keywords: Non-Newtonian metric space; compatible maps; weakly compatible maps; property E.A.; CLRT property.

1 Introduction

The dawn of the fixed point theory starts when in 1912 Brouwer proved a fixed point result for continuous self maps on a closed ball. In 1922, Banach [1] gave a very useful result known as the Banach Contraction Principle. After which a lot of implications of Banach contraction came into existence ([2-5]).

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A major shift in the arena of fixed point theory came in 1976 when Jungck [6], defined the concept of commutative maps and proved the common fixed point results for such maps. After which, Sessa [7] introduced the concept of weakly compatible, and Jungck ([8,9]) introduced the concepts of compatibility and weak compatibility. In 2002, Aamri and Moutawakil [10] introduced a generalisation of non compatible maps as property E.A. Recently, Sintuanavarat and Kumam [11] introcuced the "common limit in the range of g (i.e. CLRg)" property. Certain altercations of commutativity and compatibility can also be found in [12-16].

The study of non-Newtonian calculi has been started in 1972 by Grossman and Katz [17]. These provide an alternative to the classical calculus and they include the geometric, anageometric and bigeometric calculi, etc. In 2002 Cakmac and Basar [18], have introduced the concept of non-Newtonian metric space. Recently, Binba¸sıo˘glu, et al. [19] discussed some topological properties of the non-Newtonian metric space and also introduced the contraction principle in non-Newtonian metric space.

2 Preliminaries

A *generator* is defined as an injective map $\alpha: \mathbb{R} \to \mathbb{R}$ where the range is a subset of $\mathbb{R}$. The necessary and sufficient condition that each generator generates one arithmetic is each arithmetic is generated by one generator.

Let α be an exponential function defined as

$$\alpha: \mathbb{R} \to \mathbb{R}^+,$$
$$x \longmapsto \alpha(x) = e^x = y,$$

where $\mathbb{R}^+$ is the set of positive real numbers.

Suppose that this function α is a generator, that is, if $\alpha = I, I(x) = x \; \forall \; x \in \mathbb{R}$, then α generates the classical arithmetic. If $\alpha = \; exp, exp(x) = e^x \forall x \in \mathbb{R}$, then α generates geometrical arithmetic.

The set $\mathbb{R}(N)$ is defined as

$$\mathbb{R}(N) \coloneqq \{\alpha(x): x \in \mathbb{R}\},$$

and $\mathbb{R}(N)$ is said to be the set of non-Newtonian real numbers.

All concepts of α-arithmetic have similar properties in classical arithmetic. α-zero, α-one and other α-integers are formed as

$$\ldots, \alpha(-1), \alpha(0), \alpha(1), \ldots.$$

Let α be any generator with range A. Then, the operations α-addition, α -subtraction, α -multiplication, α -division and α -order are defined in the following way for all, $x, y \in \mathbb{R}$, respectively:

α-addition	$x \dot{+} y = \alpha\{\alpha^{-1}(x) + \alpha^{-1}(y)\}$,
α–subtraction	$x \dot{-} y = \alpha\{\alpha^{-1}(x) - \alpha^{-1}(y)\}$,
α –multiplication	$x \dot{\times} y = \alpha\{\alpha^{-1}(x) \times \alpha^{-1}(y)\}$,
α –division	$x \dot{/} y = \alpha\{\alpha^{-1}(x) \div \alpha^{-1}(y)\}$,
α –order	$x \dot{<} y \Leftrightarrow \alpha(x) < \alpha(y)$.

The fundamental properties that are provided in the classical calculus are also provided in non-Newtonian calculus, too.

For $x \in A \subset \mathbb{R}(N)$, a number α -square is described by $x \dot{\times} x$ and denoted by x^{2_N}. The symbol $\sqrt{x}^N$ denotes

$$t = \alpha\left\{\sqrt{\alpha^{-1}(x)}\right\}$$

which is the unique α nonnegative number whose α -square is equal to x and which means $t^{2_N} = x$, for each α nonnegative number t. Throughout this paper, x^{p_N} denotes the pth non-Newtonian exponent. Thus we have

$$x^{p_N} = x^{(p-1)_N} \dot{\times} x = \alpha\{[\alpha^{-1}(x)]^p\},$$

We denote by $|x|_N$ the α -absolute value of a number $x \in A \subset \mathbb{R}(N)$ defined as $\alpha(|\alpha^{-1}(x)|)$ and also

$$\sqrt{x^{2_N}}^N = |x|_N = \alpha\{|\alpha^{-1}(x)|\}$$

Thus,

$$|x|_N = \begin{cases} x, & x \dot{>} \alpha(0), \\ \alpha(0), & x = \alpha(0), \\ \alpha(0) \dot{-} x, & x \dot{<} \alpha(0). \end{cases}$$

For $x_1, x_2 \in A \subseteq \mathbb{R}(N)$, the non-Newtonian distance $|\cdot|_N$ is defined as

$$|x_1 \dot{-} x_2|_N = \alpha\{|\alpha^{-1}(x_1) - \alpha^{-1}(x_2)|\}.$$

This distance is commutative; i.e., $|x_1 \dot{-} x_2|_N = |x_2 \dot{-} x_1|_N$.

Take any $z \in \mathbb{R}(N)$, if $z \dot{>} \alpha(0)$, then z is called a positive non-Newtonian real number; if $z \dot{<} \alpha(0)$, then z is called a non-Newtonian negative real number and if $z = \alpha(0)$, then z is called an unsigned non-Newtonian real number. Non-Newtonian positive real numbers are denoted by $\mathbb{R}^+(N)$ and non-Newtonian negative real numbers by $\mathbb{R}^-(N)$ [18].

Proposition 2.1. [19]. The triangle inequality with respect to non-Newtonian distance $|\cdot|_N$, for any $x, y \in \mathbb{R}(N)$ is given by $|x \dot{+} y|_N \leq |x|_N \dot{+} |y|_N$.

Definition 2.2. [19]. Let $X \neq \emptyset$ be a set. If a function $d_N: X \times X \to \mathbb{R}^+(N)$ satisfies the following axioms for all $x, y, z \in X$:

(NM1) $d_N(x, y) = \alpha(0) = \dot{0}$ if and only if $x = y$,
(NM2) $d_N(x, y) = d_N(y, x)$,
(NM3) $d_N(x, y) \dot{\leq} d_N(x, z) \dot{+} d_N(z, y)$,

Then it is called a non-Newtonian metric on X and the pair (X, d_N) is called a non-Newtonian metric space.

Definition 2.3. [1] Let (X, d_N) be a non-Newtonian metric space, $x \in X$ and $\varepsilon \dot{>} \dot{0}$, we now define a set $B_\varepsilon^N(x) = \{y \in X : d_N(x, y) \dot{<} \varepsilon\}$, which is called a non-Newtonian open ball of radius ε with center x. Similarly, one describes the non-Newtonian closed ball as $\bar{B}_\varepsilon^N(x) = \{y \in X : d_N(x, y) \dot{\leq} \varepsilon\}$.

Example 2.4. Consider the non-Newtonian metric space $(\mathbb{R}^+(N), d_N^*)$. From the definition of d_N^*, we can verify that the non-Newtonian open ball of radius $\varepsilon \dot{<} \dot{1}$ with center x_0 appears as $(x_0 \dot{-} \varepsilon, x_0 \dot{+} \varepsilon) \subset \mathbb{R}^+(N)$.

Definition 2.5. [1] Let (X, d_X^N) and (Y, d_Y^N) be two non-Newtonian metric spaces and let $f : X \to Y$ be a function. If f satisfies the requirement that, for every $\varepsilon \dot{>} \dot{0}$, there exists $\delta \dot{>} \dot{0}$ such that $f(B_\delta^N(x)) \subset B_\varepsilon^N(f(x))$, then f is said to be non-Newtonian continuous function at $x \in X$.

Example 2.6. Given a non-Newtonian metric space (X, d_N), define a non Newtonian metric on $X \times X$ by $p((x_1, x_2), (y_1, y_2)) = d_N(x_1, y_1) \dot{+} d_N(x_2, y_2)$. Then the non-Newtonian metric $d_N : X \times X \to (\mathbb{R}^+(N), |\cdot|_N)$ is non-Newtonian(nN) continuous function on $X \times X$. To show this, let $(y_1, y_2), (x_1, x_2) \in X \times X$.

Since we have $|d_N(y_1, y_2) \dot{-} d_N(x_1, x_2)|_N \dot{\leq} d_N(x_1, y_2) \dot{+} d_N(x_2, y_2)$, it is clear that d_N is non-Newtonian continuous function on $X \times X$. Now, we emphasize on some properties of convergent sequences in a non Newtonian metric space.

Definition 2.7. [1] A sequence (x_n) in a metric space $X = (X, d_N)$ is said to be convergent if for every given $\varepsilon \dot{>} \dot{0}$ there exist an $n_0 = n_0(\varepsilon) \in N and\ x \in X$ such that $d_N(x_n, x) \dot{<} \varepsilon$ for all $n > n_0$, and it is denoted by ${}^N\lim_{n\to\infty} x_n = x$ or $x_n \overset{N}{\to} x$, as $n \to \infty$.

Definition 2.8. [1] A sequence (x_n) in a non-Newtonian metric space $X = (X, d_N)$ is said to be non-Newtonian Cauchy if for every $\varepsilon \dot{>} \dot{0}$ there exists an $n_0 = n_0(\varepsilon) \in N$ such that $d_N(x_n, x_m) \dot{<} \varepsilon$ for all $m, n > n_0$. Similarly, if for every non-Newtonian open ball $B_\varepsilon^N(x)$, there exists a natural number n_0 such that $n > n_0$, $x_n \in B_\varepsilon^N(x)$, then the sequence (x_n) is said to be non-Newtonian convergent to x.

The space X is said to be non-Newtonian complete if every non-Newtonian Cauchy sequence in X converges [19].

Proposition 2.9. [19] Let $X = (X, d_N)$ be a non-Newtonian metric space. Then

(i) a convergent sequence in X is bounded and its limit is unique,
(ii) a convergent sequence in X is a Cauchy sequence in X.

Lemma 2.10. [19] Let (X, d_N) be a non-Newtonian metric space, (x_n) a sequence in X and $x \in X$. Then $x_n \overset{N}{\to} x\ (n \to \infty)$ if and only if $d_N(x_n, x) \overset{N}{\to} \dot{0}\ (n \to \infty)$.

Lemma 2.11. [1] Let (X, d_N) be a non-Newtonian metric space and let (x_n) be a sequence in X. If the sequence (x_n) is non-Newtonian convergent, then the non-Newtonian limit point is unique.

Theorem 2.12. [1] Let (X, d_N^X) and (Y, d_N^Y) be two non-Newtonian metric spaces, $f : X \to Y$ a mapping and (x_n) any sequence in X. Then f is non-Newtonian continuous at the point $x \in X$ if and only if $f(x_n) \overset{N}{\to} f(x)$ for every sequence (x_n) with $x_n \overset{N}{\to} x\ (n \to \infty)$.

Definition 2.13. [1] Let X be a set and T a map from X to X. A fixed point of T is a point $x \in X$ such that $Tx = x$. In other words, a fixed point of T is a solution of the functional equation $Tx = x, x \in X$.

Definition 2.14. [1] Suppose that (X, d_N) is a non-Newtonian complete metric space and $T : X \to X$ is any mapping. The mapping T is said to satisfy a non-Newtonian Lipschitz condition with $k \in \mathbb{R}(N)$ if $d_N(T(x), T(y)) \dot{\leq} k \dot{\times} d_N(x, y)$ holds for all $x, y \in X$.

If $k \dot{<} \dot{1}$, then T is called a non-Newtonian contraction mapping.

Theorem 2.15. [1] Let T be a non-Newtonian contraction mapping on a non Newtonian complete metric space X. Then T has a unique fixed point.

Aamri and Moutawakil [10] introduced weakly compatible maps with E.A. property, defined as follows:

Defintion 2.16: Let S and T be two self maps of a metric space (X, d). The pair (S, T) is said to satisfy the property (E.A.) if there exists a sequence $\{x_n\}$ in X such that, $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$.

Sintunavarat and Kumam [11] introduced the concept of "Common Limit in the Range of g" property (i.e. CLRg property) and defined it as follows:

Definition 2.17: Suppose (X, d) is a metric space and $f, g: X \to X$ are two mappings then$f$ and g are said to satisfy the common limit in the range of g property if there exists a sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx$ for some $x \in X$.

3 Main Results

3.1 Compatible maps

Theorem 3.1.1: Let (X, d_N) be a complete non-Newtonian metric space, and S and T be two self maps on X, satisfying following conditions:

(i) $T(X) \subset S(X)$
(ii) S or T is continuous
(iii) $d_N(Sx, Sy) \dot{\leq} \mu \dot{\times} \max\{d_N(Tx, Ty), d_N(Tx, Sx), d_N(Tx, Sy), d_N(Ty, Sx), d_N(Ty, Sy)\}$ for all $x, y \in X$ and $\mu \in \left(\dot{0}, \alpha\left(\frac{1}{2}\right)\right)$.

Then S and T have a unique fixed point in X, provided S and T are compatible maps.

Proof: Let x_0 be an arbitrary point in X. Now by condition (i), we can choose $x_1 \in X$, such that $Sx_0 = Tx_1$. In general, we can choose x_{n+1}, such that $y_n = Sx_n = Tx_{n+1}, n = 1,2,3, ...$

From (iii), we have

$$d_N(Sx_n, Sx_{n+1}) \dot{\leq} \mu \dot{\times} \max\left\{\begin{matrix} d_N(Tx_n, Tx_{n+1}), d_N(Tx_n, Sx_n), \\ d_N(Tx_n, Sx_{n+1}), d_N(Tx_{n+1}, Sx_n), d_N(Tx_{n+1}, Sx_{n+1}) \end{matrix}\right\}$$
$$d_N(y_n, y_{n+1}) \dot{\leq} \mu \dot{\times} \max\{d_N(y_{n-1}, y_n), d_N(y_{n-1}, y_n), d_N(y_{n-1}, y_{n+1}), d_N(y_n, y_n), d_N(y_n, y_{n+1})\}$$
$$\dot{\leq} \mu \dot{\times} d_N(y_{n-1}, y_{n+1}) \dot{\leq} \alpha(2) \dot{\times} \mu \dot{\times} d_N(y_{n-1}, y_n)$$

Clearly, since $\mu \in \left(\dot{0}, \alpha\left(\frac{1}{2}\right)\right)$, the sequence is a Cauchy sequence.

Therefore, by completeness of the space X, the sequence $\{y_n\}$, converges to some point t in X. We have $\lim_{n\to\infty} y_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_{n+1} = t$. Since either S or T is continuous, we can assume that T is continuous. So we conclude that, $\lim_{n\to\infty} TSx_n = \lim_{n\to\infty} TTx_{n+1} = Tt$. Further, S and T are compatible, therefore, $\lim_{n\to\infty} d_N(STx_n, TSx_n) = 0$ implies that $\lim_{n\to\infty} STx_n = Tt$.

Now, we claim that, $Tt = t$. We have

$$d_N(STx_n, Sx_n) \dot{\leq} \mu \dot{\times} \max\left\{\begin{matrix} d_N(TTx_n, Tx_n), d_N(TTx_n, STx_n), d_N(TTx_n, Sx_n), \\ d_N(Tx_n, STx_n), d_N(Tx_n, Sx_n) \end{matrix}\right\}$$

Taking limit as $n \to \infty$, and using condition (iii), we have

$d_N(Tt, t) \dot{\leq} \mu \dot{\times} d_N(Tt, t)$ implies that $d_N(Tt, t)\left(\dot{1} \dot{-} \mu\right) \dot{\leq} \dot{0} \Rightarrow Tt = t$.

Next, we will show that $St = Tt = t$. For this purpose, we take, $x = x_n$, $y = t$ in (iii), we have, $d_N(Sx_n, St) \dot{\leq} \mu \dot{\times} \max\{d_N(Tx_n, Tt), d_N(Tx_n, Sx_n), d_N(Tx_n, St), d_N(Tt, St), d_N(Tt, Sx_n)\}$

Taking limit as $n \to \infty$, we have, $d_N(t, St) \dot{\leq} \mu \dot{\times} d_N(t, St)$, i.e., $t = St$.

Thus, t is the common fixed point of S and T.

We now assume that, $u(\neq t)$ is another fixed point of S and T. Then,

$$d_N(t, u) = d_N(St, Su) \dot{\leq} \mu \dot{\times} \max\{d_N(Tt, Tu), d_N(Tt, St), d_N(Tt, Su), d_N(Tu, St), d_N(Tu, Su)\}$$

which implies that, $d_N(t, u) \dot{\leq} \mu \dot{\times} d_N(t, u)$. So, $t = u$.

So, S and T have a unique common fixed point.

3.2 Weakly compatible maps

Theorem 3.2.1: Let, S and T are weakly compatible self maps of a non-Newtonian metric space (X, d_N) satisfying:

(i) $S(X) \subseteq T(X)$
(ii) Any one of the subspace $S(X)$ or $T(X)$ is complete.
(iii) $d_N(Sx, Sy) \dot{\leq} \mu \dot{\times} \max\{d_N(Tx, Ty), d_N(Tx, Sx), d_N(Tx, Sy), d_N(Ty, Sx), d_N(Ty, Sy)\}$ for all $x, y \in X$ and $\dot{0} \dot{<} \mu \dot{<} \alpha\left(\frac{1}{2}\right)$.

Then, S and T have a unique fixed point in X.

Proof: From theorem 3.1.1, we conclude that, $\{x_n\}$ is a non-Newtonian Cauchy sequence in X. Since, either $S(X)$ or $T(X)$ is complete, for definiteness assume that $T(X)$ is complete subspace of X, then the subsequence of $\{x_n\}$ must converge to a limit in $T(X)$, say t. Let, $u = T^{-1}t$. Then, $Tu = t$, as $\{x_n\}$ is a non-Newtonian Cauchy sequence containing a convergent subsequence, thereby the convergence of subsequence of the convergent sequence. Now, we show that $Su = t$.

From condition (iii), we have,

$$d_N(Sx_n, Su) \dot{\leq} \mu \dot{\times} \max\{d_N(Tx_n, Tu), d_N(Tx_n, Sx_n), d_N(Tx_n, Su), d_N(Tu, Sx_n), d_N(Tu, Su)\}$$

Taking limit as $n \to \infty$, we have,

$$d_N(t, Su) \dot{\leq} \mu \dot{\times} d_N(t, Su) \Rightarrow d_N(t, Su) = \dot{0}$$

Hence, $Su = Tu = t$. Thus, u is a coincidence point of S and T. Since, S and T are weakly compatible, it shows that $STu = TSu$, i.e., $St = Tt$.

Now, suppose that, $St \neq t$, therefore, $d_N(St, t) \dot{>} \dot{0}$. Now, from (iii), we have,

$$d_N(St, Su) \dot{\leq} \mu \dot{\times} \max\left\{\begin{matrix} d_N(Tt, Tu), d_N(Tt, St), d_N(Tt, Su), \\ d_N(Tu, St), d_N(Tu, Su) \end{matrix}\right\}$$

So, we have,

$$d_N(St, t) \dot{\leq} \mu \dot{\times} \max\{d_N(Tt, t), d_N(St, t)\} = \mu \dot{\times} d_N(St, t)$$

which is a contradiction as $\mu \in \left(\dot{0}, \alpha\left(\frac{1}{2}\right)\right)$. Hence, $St = t = Tt$. And hence t is a common fixed point of S and T. Uniqueness follows easily. So, S and T have a unique common fixed point.

3.3 E.A. Property in non Newtonian metric spaces

Defintion 3.3.1: Let S and T be two self maps of a non- Newtonian metric space (X, d_N). The pair (S, T) is said to satisfy the property (E.A.) if there exists a sequence $\{x_n\}$ in X such that, $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$.

Now, we prove a common fixed point theorem for the maps satisfying (E.A.) property in non-Newtonian metric spaces.

Theorem 3.3.2: Let, S and T be self maps on a non Newtonian Metric space (X, d_N), satisfying the conditions:

(i) $d_N(Sx, Sy) \dot{\leq} \mu \dot{\times} d_N(Tx, Ty)$ for all $x, y \in X$ and $\mu \in (\dot{0}, \dot{1})$
(ii) S and T satisfy the property (E.A.)
(iii) $T(X)$ is a closed subspace of X.

Then S and T have a unique fixed point in X, provided S and T are weakly compatible maps.

Proof: Since S and T satisfy (E.A.) property, there exists a sequence $\{x_n\}$ in X such that, $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$. Since $T(X)$ is a closed subspace of X, there exists an $a \in T(X)$ such that $\lim_{n\to\infty} T(x_n) = Ta = t = \lim_{n\to\infty} S(x_n)$.

We claim that $Sa = Ta = t$.

By condition (i) $d_N(Sx_n, Sa) \dot{\leq} \mu \dot{\times} d_N(Tx_n, Ta)$.

Taking limit $n \to \infty$, $d_N(t, Sa) \dot{\leq} \dot{0}$ which implies that $Sa = t = Ta$. So, a is a coincidence point of S and T. Since S and T are weakly compatible, it follows that $STa = TSa$.

We can therefore infer

$$d_N(Sa, SSa) \dot{\leq} \mu \dot{\times} d_N(Ta, TSa) = \mu \dot{\times} d_N(Sa, STa) = \mu \dot{\times} d_N(Sa, SSa).$$

Hence, $d_N(Sa, SSa) \dot{\times} (1 - \mu) \dot{\leq} \dot{0}$. Since $\mu \in (\dot{0}, \dot{1})$, $Sa = SSa = TSa$, i.e. Sa is common fixed point of S and T.

To see that S and T can have only one common fixed point, suppose that $x = Tx = Sx$ and $y = Ty = Sy$.

Then (i) implies that $d_N(x, y) = d_N(Sx, Sy) \dot{\leq} \mu \dot{\times} d_N(Tx, Ty) = \mu \dot{\times} d_N(x, y)$, or $d_N(x, y) \dot{\times} (\dot{1} \dot{-} \mu) \dot{\leq} \dot{0}$. Since, $\mu \dot{<} \dot{1}$, $x = y$.

So, S and T have a unique common fixed point.

Theorem 3.3.3: Let, S and T be self maps on a non Newtonian Metric space (X, d_N), satisfying the conditions:

(i) $d_N(Sx, Sy) \dot{\leq} \mu \dot{\times} \max\left\{\begin{matrix} d_N(Tx, Ty), d_N(Tx, Sx), d_N(Tx, Sy), \\ d_N(Ty, Sx), d_N(Ty, Sy) \end{matrix}\right\}$ for all $x, y \in X$ and $\mu \in \left(\dot{0}, \alpha\left(\frac{1}{2}\right)\right)$,
(ii) S and T satisfy the property (E.A.),
(iii) $T(X)$ is a closed subspace of X.

Then S and T have a unique fixed point in X, provided S and T are weakly compatible maps.

Proof: Since S and T satisfy (E.A.) property, there exists a sequence $\{x_n\}$ in X such that, $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$. Since $T(X)$ is a closed subspace of X, there exists an $a \in T(X)$ such that $\lim_{n\to\infty} T(x_n) = Ta = t = \lim_{n\to\infty} S(x_n)$.

We claim that $Sa = Ta = t$.

By condition (i)

$$d_N(Sx_n, Sa) \dot{\leq} \mu \dot{\times} \max\begin{Bmatrix} d_N(Tx_n, Ta), d_N(Tx_n, Sx_n), d_N(Tx_n, Sa), \\ d_N(Ta, Sx_n), d_N(Ta, Sa) \end{Bmatrix}$$

Taking limit as $n \to \infty$, $d_N(t, Sa) \dot{\leq} \mu \dot{\times} d_N(t, Sa)$ which implies that $Sa = t = Ta$. So, a is a coincidence point of S and T. Since S and T are weakly compatible, it follows that $STa = TSa$.

We can therefore infer

$$\begin{aligned} d_N(Sa, SSa) &\dot{\leq} \mu \dot{\times} \max\begin{Bmatrix} d_N(Ta, TSa), d_N(Ta, Sa), d_N(Ta, SSa), \\ d_N(TSa, Sa), d_N(TSa, SSa) \end{Bmatrix} = \mu \dot{\times} d_N(Sa, STa) \\ &= \mu \dot{\times} d_N(Sa, SSa). \end{aligned}$$

Hence, $d_N(Sa, SSa)(1-\mu) \dot{\leq} \dot{0}$. Since $\mu \in \left(\dot{0}, \alpha\left(\frac{1}{2}\right)\right)$, $Sa = SSa = TSa$, i.e. Sa is common fixed point of S and T.

To see that S and T can have only one common fixed point, suppose that $x = Tx = Sx$ and $y = Ty = Sy$.

Then (i) implies that

$$d_N(x, y) = d_N(Sx, Sy) \dot{\leq} \mu \dot{\times} \max\begin{Bmatrix} d_N(Tx, Ty), d_N(Tx, Sx), d_N(Tx, Sy), \\ d_N(Ty, Sx), d_N(Ty, Sy) \end{Bmatrix} = \mu \dot{\times} d_N(x, y) \quad , \quad \text{or}$$
$$d_N(x, y) \dot{\times} \left(\dot{1} \dot{-} \mu\right) \dot{\leq} \dot{0}. \text{ Since, } \mu \dot{<} \dot{1}, x = y.$$

So, S and T have a unique common fixed point.

Remark 3.3.4: It can be easily observed that, the mappings which satisfy the property (E.A.) need not satisfy the condition of containment of range of one mapping into another, which is necessary for weakly compatible maps. Moreover, the condition of continuity is not required for the containment of range, minimizes the commutativity condition to commutativity at their coincidence points. Also, completeness requirement of the space is replaced with a more natural condition of closedness of the range.

3.4 (CLRT) property in non Newtonian metric space

Definition 3.4.1: Suppose (X, d) be a metric space and $S, T: X \to X$. The mappings S and T are said to satisfy the common limit in the range of T property if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = Tx \text{ for some } x \in X.$$

Theorem 3.4.2: Let, S and T be self maps of a non Newtonian metric space (X, d_N) satisfying

(i) $d_N(Sx, Sy) \dot{\leq} \mu d_N(Tx, Ty)$ for all $x, y \in X$ and $\mu \in (\dot{0}, \dot{1})$
(ii) $CLRT$ property

Then S and T have a unique fixed point in X, provided S and T are weakly compatible.

Proof: Since S and T satisfy $CLRT$ property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = Ta = t \text{ for some } a, t \in X.$$

Now, from condition (i) we have $d_N(Sx_n, Sa) \dot{\leq} \mu d_N(Tx_n, Ta)$

Taking limit as $n \to \infty$, $d_N(t, Sa) \dot{\leq} \dot{0}$ which implies that $Sa = t = Ta$. Hence a is a coincidence point of S and T. Since S and T are weakly compatible, it follows that $STa = TSa$.

We can therefore infer

$$d_N(Sa, SSa) \dot{\leq} \mu d_N(Ta, TSa) = \mu d_N(Sa, STa) = \mu d_N(Sa, SSa).$$

Hence, $d_N(Sa, SSa) \dot{\times} (1 - \mu) \dot{\leq} \dot{0}$. Since $\mu \in (\dot{0}, \dot{1})$, $Sa = SSa = TSa$, i.e. $S(a)$ is common fixed point of S and T.

To see that S and T can have only one common fixed point, suppose that $x = Tx = Sx$ and $y = Ty = Sy$.

Then (i) implies that $d_N(x, y) = d_N(Sx, Sy) \dot{\leq} \mu d_N(Tx, Ty) = \mu d_N(x, y)$, or $d_N(x, y) \dot{\times} (\dot{1} \dot{-} \mu) \dot{\leq} \dot{0}$. Since, $\mu \dot{<} \dot{1}, x = y$.

So, S and T have a unique common fixed point.

Theorem 3.4.3: Let, S and T be self maps of a non Newtonian metric space (X, d_N) satisfying

(i) $d_N(Sx, Sy) \dot{\leq} \mu \dot{\times} \max\{d_N(Tx, Ty), d_N(Tx, Sx), d_N(Tx, Sy), d_N(Ty, Sx), d_N(Ty, Sy)\}$ for all $x, y \in X$ and $\mu \in \left(\dot{0}, \alpha\left(\frac{1}{2}\right)\right)$,

(ii) $CLRT$ property

Then S and T have a unique fixed point in X, provided S and T are weakly compatible.

Proof: Since S and T satisfy $CLRT$ property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = Ta = t \text{ for some } a, t \in X.$$

Now, from condition (i) we have

$$d_N(Sx_n, Sa) \dot{\leq} \mu \dot{\times} \max\{d_N(Tx_n, Ta), d_N(Tx_n, Sx_n), d_N(Tx_n, Sa), d_N(Ta, Sx_n), d_N(Ta, Sa)\}$$

Taking limit as $n \to \infty$, $d_N(t, Sa) \dot{\leq} \dot{0}$ which implies that $Sa = t = Ta$. Hence a is a coincidence point of S and T. Since S and T are weakly compatible, it follows that $STa = TSa$.

We can therefore infer

$$\begin{aligned} d_N(Sa, SSa) &\dot{\leq} \mu \dot{\times} \max\{d_N(Ta, TSa), d_N(Ta, Sa), d_N(Ta, SSa), d_N(TSa, Sa), d_N(TSa, SSa)\} \\ &= \mu \dot{\times} d_N(Sa, STa) = \mu \dot{\times} d_N(Sa, SSa). \end{aligned}$$

Hence, $d_N(Sa, SSa) \dot{\times} (1 - \mu) \dot{\leq} \dot{0}$. Since $\mu \in \left(\dot{0}, \alpha\left(\frac{1}{2}\right)\right)$, $Sa = SSa = TSa$, i.e. $S(a)$ is common fixed point of S and T.

To see that S and T can have only one common fixed point, suppose that $x = Tx = Sx$ and $y = Ty = Sy$.

Then (i) implies that

$$d_N(x,y) = d_N(Sx,Sy) \dot{\leq} \mu \dot{\times} \max\left\{\begin{matrix} d_N(Tx,Ty), d_N(Tx,Sx), d_N(Tx,Sy), \\ d_N(Ty,Sx), d_N(Ty,Sy) \end{matrix}\right\} = \mu \dot{\times} d_N(x,y) \quad , \quad \text{or}$$

$x = y$.

So, S and T have a unique common fixed point.

Competing Interests

Authors have declared that no competing interests exist.

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