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The Gamma Function and Its Analytical Applications

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Authors' contributions

This work was carried out in collaboration between both authors. Authors MMI and KIT designed the study together. Author KIT did the preliminary analysis and wrote the first draft of the manuscript. Author MMI re-organized the draft and did further analysis. Both authors read and approved the final manuscript.

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Short Communication

Abstract

This paper explores the history and properties of the Gamma function with some analytical applications. Specifically, the Gamma function is employed to prove the legitimacy of the Standard Normal Distribution and for evaluation of some integrals involving the Laplace and Fourier Transforms using very simple techniques. Moreover, this paper demonstrates that the Gamma function is not a mere formula and proof in itself but rather an essential tool for applications in evaluating integrals that occur in practice and also in simplifying proofs of some other important identities and theorems in mathematics.

Keywords: Gamma function; applications; standard normal distribution; Laplace and Fourier transforms.

1 Introduction

Many special functions arise in the consideration of the solutions of several differential equations. Special functions are some essential functions that are important enough to be given their own name. These include the well-known logarithmic, exponential and trigonometric functions, and extend to cover the Gamma, beta and zeta functions, spherical and parabolic cylinder functions, and the class of orthogonal polynomials,

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among many others. The vast field of these functions contains many formulae and identities used by mathematicians, engineers and physicists. Special functions have extensive applications in pure mathematics, as well as in applied areas such as acoustics, electrical current, fluid dynamics, heat conduction, solutions of wave equations, moments of inertia and quantum mechanics [1-3]. At the heart of the theory of special functions lies the Gamma function, in that nearly almost all of the classical special functions can be evaluated by this powerful function. Gamma functions have explicit series and integral functional representations, and thus provide ideal tools for establishing useful products and transformation formulae. In addition, applied problems frequently require solutions of a function in terms of parameters, rather than merely in terms of a variable, and such a solution is perfectly provided for by the parametric nature of the Gamma function. As a result, the Gamma function can be used to evaluate physical problems in diverse areas of applied mathematics. While the Gamma function's original intent was to model and interpolate the factorial function, mathematicians and geometers have discovered and developed many other interesting applications thus playing a particularly useful role in applied mathematics. Equations involving Gamma functions are of great interest to mathematicians and scientists, and newly proven identities for these functions assist in finding solutions for many differential and integral equations. There exist a vast number of such identities, representations and transformations for the Gamma function, the comprehensive text [4] providing over 400 integral and series representations for these functions. Gamma functions thus provide a rich field for ongoing research, which continues to produce new results. In 1959, in [5], It was stated that "of the so-called 'higher mathematical functions', the Gamma function is undoubtedly the most fundamental". For instance the rising factorial provides a direct link between the Gamma and hyper-geometric functions, and most hyper-geometric identities can be more elegantly expressed in terms of the Gamma function. In [6], it is stated clearly that, "the Gamma function and beta integrals are essential to understanding hyper-geometric functions." It is thus enlightening and rewarding to explore the various representations and relations of the Gamma function.

The aim of this paper is to trace a brief history of the development of the Gamma function and illustrate its applications in the proofs of some useful properties and identities in mathematics and statistics.

2 Materials and Methods

This section presents some very relevant definitions for this work.

Definition 1: (The Gamma function)

The Gamma function, $\Gamma(\beta)$ commonly referred to as Euler's integral is defined as

$$\Gamma(\beta) = \int_0^\infty t^{\beta-1} e^{-t}\, dt, \qquad \beta > 0. \tag{1.1}$$

See [7-9] for further discussions on Gamma functions.

The central relation is given (see [10]) as

$$\Gamma(\beta+1) = \beta\,\Gamma(\beta) = \beta\,!, \quad \beta > 0. \tag{1.2}$$

Now splitting the integral (1.1), for $x \geq 0$, yields two incomplete Gamma functions:

$$\gamma(\beta, x) = \int_0^x t^{\beta-1} e^{-t} dt\ ,$$

$$\Gamma(\beta, x) = \int_x^\infty t^{\beta-1} e^{-t} dt$$

Prym was the first to investigate these functions in 1877 and so $\Gamma(\beta, x)$ has been called Prym's functions or sometimes referred to as the complementary incomplete Gamma functions.

Note that

$$\Gamma(\beta, 0) = \Gamma(\beta)$$

and

$$\gamma(\beta, x) + \Gamma(\beta, x) = \Gamma(\beta)$$

for all $x \geq 0$ and $\beta > 0$. Also

$$\Gamma(1, x) = e^{-x} \text{ and } \gamma(1, x) = 1 - e^{-x}.$$

The integral $\Gamma(\beta, x)$ converges for real β and the integral $\gamma(\beta, x)$ converges for $\beta > 0$ for all $x > 0$. (See [11] for further details on Incomplete Gamma Functions.)

Another very important topic considered in this paper is the standard normal distribution (Fig. 1) which is a special type of the normal distribution with mean equal to zero and the variance equal to one **(**$Z \sim N(0,1)$**)**.

Definition 2: Let Z be an absolutely continuous random variable. Then Z is said to have a standard normal distribution if its probability density function is given as

$$f(z) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2} z^2\right], \quad -\infty < z < \infty. \tag{1.3}$$

(See [12, p.167]). Then by (1.3) we have

$$\int_{-\infty}^{\infty} f(z) dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} dz = 1. \tag{1.4}$$

The integral (1.4) is the total area bounded by the curve of the standard normal distribution and the horizontal axis which is equal to 1 (Fig. 1).

Remark 1

The property (1.4) means that the probability of the entire sample space occurring is certain. Once this property is satisfied for a probability distribution then the distribution is said to be legitimate.

Definition 3: (Moment Generating Function)

The moment generating function of a standard normal random variable Z is defined for any $t \in \Re$ by

$$M_Z(t) = E\left(e^{tz}\right) = e^{\frac{1}{2}t^2}. \tag{1.5}$$

See [12] and the references therein.

Definition 4: **(**Laplace Transforms)

The Laplace transform of a function $f:[0,\infty)\to\Re$ denoted by $L[f](\alpha)$ is defined as

$$\begin{aligned} L[f](\alpha) &= \int_0^{\infty} f(t)\, e^{-\alpha t} dt \\ F(\alpha) &= \lim_{A\to\infty}\int_0^{A} f(t) e^{-\alpha t} dt \end{aligned} \tag{1.6}$$

where $\alpha\in\Re$ is a parameter of the transform. See [13].

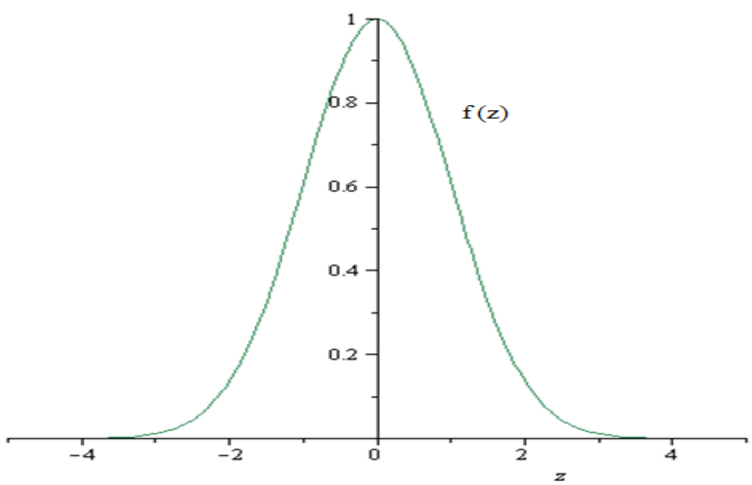

Fig. 1. The standard normal distribution curve

Remark 2

The defining equation for the Laplace transform is an improper integral. The existence of the Laplace transform of f depends upon the existence of the limit. If the limit exist then the integral (1.6) is said to converge. If the limit does not exist then the integral is said to diverge and in that case there is no Laplace transform for *f*.

Another consideration is the Fourier transform of a function $f:[0,\infty)\to C$. This can be defined in several ways. In our case, we consider the integral representation.

Definition 5 : (Fourier transforms as integrals)

Let $f:\Re\to C$. The Fourier transform of $f\in L^1(R)$, denoted by $F[f](\cdot)$, is defined by

$$F[f](t) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{-(ixt)} dx \tag{1.7}$$

for $t\in R$ for which the integral exists. (See [14]).

3 Results and Discussion

The main focus of this paper is to present some applications of the Gamma function in the evaluation of some integrals. Here, we demonstrate the use of the Gamma function in the proof of the legitimacy of the standard normal distribution as a probability density function; in finding the Laplace transform and the Fourier transform of continuous functions on $[0,\infty)$.

3.1 The standard normal distribution

This section presents a clear proof of a special property of the normal distribution called the legitimacy of the standard normal distribution. In this proof, the concept of the Gamma function is applied. Thus a step by step approach is used to write the standard normal distribution in the form of the Gamma function.

We begin the proof of (1.4) as follows:

Let the probability density function be

$$f(z) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}z^2\right], \text{ where } -\infty < z < \infty. \tag{1.8}$$

Then

$$\begin{aligned}
\int_{-\infty}^{\infty} f(z)dz &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} dz \\
&= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} e^{-\frac{1}{2}z^2} dz + \int_{0}^{\infty} e^{-\frac{1}{2}z^2} dz \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(-\int_{0}^{-\infty} e^{-\frac{1}{2}z^2} dz + \int_{0}^{\infty} e^{-\frac{1}{2}z^2} dz \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\int_{0}^{\infty} e^{-\frac{1}{2}z^2} dz + \int_{0}^{\infty} e^{-\frac{1}{2}z^2} dz \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(2\int_{0}^{\infty} e^{-\frac{1}{2}z^2} dz \right) \\
&= \frac{\sqrt{2}}{\sqrt{\pi}} \left(\int_{0}^{\infty} e^{-\frac{1}{2}z^2} dz \right).
\end{aligned} \tag{1.9}$$

To evaluate the integral (1.9), apply change of variables and substitute for the Gamma function.

Let $u = \frac{1}{2}z^2$. Then $z = \sqrt{2u}$ implies $dz = \frac{u^{-1/2}}{\sqrt{2}} du$.

Now, when $z = 0, \quad u = 0$ and as $z \to \infty, \quad u \to \infty.$

This transforms (1.9) to

$$\int_{-\infty}^{\infty} f(z)dz = \frac{\sqrt{2}}{\sqrt{\pi}}\left(\int_0^{\infty} e^{-u}\frac{u^{-1/2}}{\sqrt{2}}du\right) = \frac{1}{\sqrt{\pi}}\left(\int_0^{\infty} e^{-u}u^{-1/2}du\right)$$
$$= \frac{1}{\sqrt{\pi}}\int_0^{\infty} u^{1/2-1}e^{-u}du$$
$$= \frac{1}{\sqrt{\pi}}\Gamma\left(\tfrac{1}{2}\right) \tag{1.10}$$

Equation (1.10) is obtained through the application in simple manner of the Gamma function (1.1).

Since $\Gamma\left(1/2\right) = \sqrt{\pi}$ [13, p.152], hence

$$\int_{-\infty}^{\infty} f(z)dz = \frac{1}{\sqrt{\pi}}\Gamma\left(1/2\right) = 1$$

as required.

3.2 Moment generating function

One of the important properties of the standard normal distribution is its moment generating function. Again, we apply the Gamma function to obtain the moment generating function of the standard normal distribution.

Definition 6

The moment generating function of a standard normal random variable Z is defined for any $t \in \Re$ as

$$M_Z(t) = E\left(e^{tz}\right) = e^{1/2\, t^2} \tag{1.11}$$

We want to establish this result (1.11) with a simple application of the Gamma function.

By definition

$$E\left(e^{tz}\right) = \int_{-\infty}^{\infty} e^{tz} f(z)dz = M_Z(t). \tag{1.12}$$

Then

$$M_Z(t) = E\left(e^{tz}\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{tz}e^{-\frac{1}{2}z^2}dz$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\left(tz-\frac{1}{2}z^2\right)}dz \qquad = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(z^2-2tz\right)}dz$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[(z-t)^2-t^2\right]}dz \quad = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t)^2}e^{\frac{1}{2}t^2}dz$$
$$= \frac{e^{\frac{1}{2}t^2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t)^2}dz \tag{1.13}$$

To evaluate the integral, apply change of variables.

Let $u = z - t$. Then $du = dz$. As $z \to -\infty, \quad u \to -\infty$ and as $z \to \infty, \quad u \to \infty$. Thus (1.13) becomes

$$M_Z(t) = \frac{e^{\frac{1}{2}t^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2} du \tag{1.14}$$

Since the normal distribution is symmetric about the origin, integrating under the entire curve is the same as integrating from zero to positive infinity and multiplying by 2. Thus (1.14) yields

$$M_Z(t) = \frac{2e^{\frac{1}{2}t^2}}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{1}{2}u^2} du \tag{1.15}$$

Applying change of variables again and substitute for the Gamma function:

Let $x = \frac{1}{2}u^2$. Then $u = \sqrt{2x}$ and $du = \frac{\sqrt{2}}{2} x^{-1/2} dx$. Now $u = 0, \quad x = 0$ and as $u \to \infty, \quad x \to \infty$. Thus, (1.15) becomes

$$\begin{aligned} M_Z(t) &= \frac{2e^{\frac{1}{2}t^2}}{\sqrt{2\pi}} \int_0^{\infty} \frac{\sqrt{2}}{2} x^{-1/2} e^{-x} dx \\ &= \frac{e^{\frac{1}{2}t^2}}{\sqrt{\pi}} \int_0^{\infty} x^{-1/2} e^{-x} dx \end{aligned}$$

$$M_Z(t) = \frac{e^{\frac{1}{2}t^2}}{\sqrt{\pi}} \int_0^{\infty} x^{1/2 - 1} e^{-x} dx$$

Hence

$$M_Z(t) = \frac{e^{\frac{1}{2}t^2}}{\sqrt{\pi}} \Gamma\left(1/2\right) = \frac{e^{\frac{1}{2}t^2}}{\sqrt{\pi}} \cdot \sqrt{\pi} = e^{\frac{1}{2}t^2}$$

as required.

3.3 Laplace transforms

Pierre Simon de Laplace was a French mathematician who lived during 1749-1827, and was essentially interested to describe nature using mathematics. The main idea behind the Laplace Transformation is that one can solve an equation (or system of equations) containing differential and integral terms by transforming the equation in "t-space" to one in "s-space". This makes the problem much easier to solve. The kind of problems where the Laplace transform is invaluably occurs are in electronics.

We will find the Laplace Transforms of some functions which are very useful when solving problems in science and engineering using the Gamma function. We demonstrate the application for three different functions and the idea can be employed for many other cases:

Proposition 1

Let f be a continuous functions on $\Re$. The following are the Laplace transforms of the given functions.

(i) $f(t)=t^n e^{at}$, $L\{f(t)\}=\dfrac{n!}{(s-a)^{n+1}}$, $\quad s>a$

(ii) $f(t)=e^{at}\sin\omega t$, $L\{f(t)\}=\dfrac{\omega}{(s-a)^2+\omega^2}$, $\quad s>a$

(iii) $f(t)=e^{at}\cos\omega t$, $L\{f(t)\}=\dfrac{s-a}{(s-a)^2+\omega^2}$, $s>a$,

where $a\in\Re$.

Proof

(i) By definition 3 we have

$$L\{t^n e^{at}\}=F(s)=\int_0^\infty t^n e^{at}\ e^{-st}dt$$

$$F(s)=\int_0^\infty t^n\ e^{-(s-a)t}dt \tag{1.16}$$

Apply change of variables.

Let $u=(s-a)t \Leftrightarrow t=\dfrac{1}{(s-a)}u.$ Then $dt=\dfrac{1}{(s-a)}du.$ Now $t=0,\ \ u=0$ and as $t\to\infty,\quad u\to\infty.$ So (1.16) becomes

$$F(s)=\int_0^\infty\left(\frac{u}{s-a}\right)^n e^{-u}\frac{1}{(s-a)}du$$

$$F(s)=\int_0^\infty\frac{1}{(s-a)^n}\cdot u^n\cdot e^{-u}\cdot\frac{1}{(s-a)}du$$

$$F(s)=\frac{1}{(s-a)^{n+1}}\int_0^\infty u^n e^{-u}du \tag{1.17}$$

We then re-arrange (1.17) in a way that we can apply the Gamma function. That is:

$$F(s)=\frac{1}{(s-a)^{n+1}}\int_0^\infty u^{(n+1)-1}e^{-u}du \tag{1.18}$$

Then substitute for the Gamma function in (1.18) to obtain

$$F(s)=\frac{1}{(s-a)^{n+1}}\Gamma(n+1)$$

$$F(s)=\frac{n!}{(s-a)^{n+1}},\ s>0$$

as required.

(ii) By definition 3 we have

$$L\{e^{at}\sin\omega t\}=F(s)=\int_0^\infty e^{at}\sin\omega t\ e^{-st}dt$$

$$F(s)=\int_0^\infty \sin\omega t\ e^{-(s-a)t}dt \tag{1.19}$$

$$F(s)=\int_0^\infty \frac{1}{2i}\left(e^{i\omega t}-e^{-i\omega t}\right)e^{-(s-a)t}dt$$

$$F(s)=\frac{1}{2i}\int_0^\infty \left\{e^{i\omega t}\ e^{-(s-a)t}-e^{-i\omega t}\ e^{-(s-a)t}\right\}dt$$

$$F(s)=\frac{1}{2i}\int_0^\infty \left\{e^{-[(s-a)-i\omega]t}-e^{-[(s-a)+i\omega]t}\right\}dt$$

$$F(s)=\frac{1}{2i}\left[\int_0^\infty e^{-[(s-a)-i\omega]t}dt-\int_0^\infty e^{-[(s-a)+i\omega]t}dt\right] \tag{1.20}$$

Apply change of variables:

Let $u=[(s-a)-i\omega]t \Leftrightarrow t=\frac{1}{[(s-a)-i\omega]}u.$ Then $dt=\frac{1}{[(s-a)-i\omega]}du.$

Also let $v=[(s-a)+i\omega]t \Leftrightarrow t=\frac{1}{[(s-a)+i\omega]}v.$ Then $dt=\frac{1}{[(s-a)+i\omega]}dv.$

When $t=0,\quad u=v=0$ and as $t\to\infty,\quad u=v\to\infty.$

Thus (1.20) becomes

$$F(s)=\frac{1}{2i}\left[\int_0^\infty e^{-u}\cdot\frac{1}{[(s-a)-i\omega]}\cdot du-\int_0^\infty e^{-v}\cdot\frac{1}{[(s-a)+i\omega]}\cdot dv\right]$$

$$F(s)=\frac{1}{2i}\left[\frac{1}{[(s-a)-i\omega]}\int_0^{\infty}e^{-u}du-\frac{1}{[(s-a)+i\omega]}\int_0^{\infty}e^{-v}dv\right]$$

$$F(s)=\frac{1}{2i}\left[\frac{1}{[(s-a)-i\omega]}\int_0^{\infty}u^{1-1}e^{-u}du-\frac{1}{[(s-a)+i\omega]}\int_0^{\infty}v^{1-1}e^{-v}dv\right]$$

$$=\frac{1}{2i}\left[\frac{1}{\left[(s-a)-i\omega\right]}\Gamma(1)-\frac{1}{\left[(s-a)+i\omega\right]}\Gamma(1)\right]$$

Since $\Gamma(1)=0!=1$, we have

$$F(s)=\frac{1}{2i}\left[\frac{1}{\left[(s-a)-i\omega\right]}-\frac{1}{\left[(s-a)+i\omega\right]}\right]$$

$$F(s)=\frac{1}{2i}\left[\frac{[(s-a)+i\omega]-[(s-a)-i\omega]}{(s-a)^2-(i\omega)^2}\right]$$

$$F(s)=\frac{1}{2i}\left[\frac{2i\omega}{(s-a)^2-(i\omega)^2}\right]=\frac{\omega}{(s-a)^2+\omega^2},\ \text{since }\left(i^2=-1\right).$$

(iii)By definition 3 we have;

$$L\left\{e^{at}\cos\omega t\right\}=F(s)=\int_0^{\infty}e^{at}\cos\omega t\ e^{-st}dt$$

$$F(s)=\int_0^{\infty}\cos\omega t\ e^{-(s-a)t}dt \tag{1.21}$$

$$F(s)=\int_0^{\infty}\frac{1}{2}\left(e^{i\omega t}+e^{-i\omega t}\right)e^{-(s-a)t}dt$$

$$F(s)=\frac{1}{2}\int_0^{\infty}\left\{e^{i\omega t}\ e^{-(s-a)t}+e^{-i\omega t}\ e^{-(s-a)t}\right\}dt$$

$$F(s)=\frac{1}{2}\int_0^{\infty}\left\{e^{-[(s-a)-i\omega]t}+e^{-[(s-a)+i\omega]t}\right\}dt$$

$$F(s)=\frac{1}{2}\left[\int_0^{\infty}e^{-[(s-a)-i\omega]t}dt+\int_0^{\infty}e^{-[(s-a)+i\omega]t}dt\right] \tag{1.22}$$

Apply change of variables:

Let $u=[(s-a)-i\omega]t\Leftrightarrow t=\frac{1}{[(s-a)-i\omega]}u.$ Then $dt=\frac{1}{[(s-a)-i\omega]}du.$

Also let $v=[(s-a)+i\omega]t \Leftrightarrow t=\frac{1}{[(s-a)+i\omega]}v.$ Then $dt=\frac{1}{[(s-a)+i\omega]}dv.$

When $t=0,\quad u=v=0$ and as $t\to\infty,\quad u=v\to\infty.$ Thus (1.22) becomes

$$F(s)=\frac{1}{2}\left[\int_0^\infty e^{-u}\cdot\frac{1}{[(s-a)-i\omega]}\cdot du+\int_0^\infty e^{-v}\cdot\frac{1}{[(s-a)+i\omega]}\cdot dv\right]$$

$$F(s)=\frac{1}{2}\left[\frac{1}{[(s-a)-i\omega]}\int_0^\infty e^{-u}du+\frac{1}{[(s-a)+i\omega]}\int_0^\infty e^{-v}dv\right]$$

$$F(s)=\frac{1}{2}\left[\frac{1}{[(s-a)-i\omega]}\int_0^\infty u^{1-1}e^{-u}du+\frac{1}{[(s-a)+i\omega]}\int_0^\infty v^{1-1}e^{-v}dv\right]$$

$$F(s)=\frac{1}{2}\left[\frac{1}{[(s-a)-i\omega]}\Gamma(1)+\frac{1}{[(s-a)+i\omega]}\Gamma(1)\right]$$

Since $\Gamma(1)=0!=1$ and $i^2=-1$, we have

$$F(s)=\frac{1}{2}\left[\frac{1}{[(s-a)-i\omega]}+\frac{1}{[(s-a)+i\omega]}\right]$$

$$F(s)=\frac{1}{2}\left[\frac{[(s-a)+i\omega]+[(s-a)-i\omega]}{(s-a)^2-(i\omega)^2}\right]\quad=\frac{s-a}{(s-a)^2+\omega^2}.$$

3.4 Fourier transform

There are several ways to define the Fourier transform of a function $f:R\to C$. In this section, we adopt the integral representation as in definition 4 and use to prove the established results with the application of the Gamma function.

We begin with the popular ***one-sided decaying exponential*** which is defined as

$$f(t)=\begin{cases}0, & t<0\\ e^{-t}, & t\geq 0\end{cases}$$

Proposition 2

The Fourier transform of the one-sided decaying exponential is

$$F[f](x)=\frac{1}{(ix+1)\sqrt{2\pi}}.$$

We prove this transformation with a simple technique associated with the Gamma function.

Proof

Write

$$F[f](x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-(ixt)}dt$$

$$F[f](x)=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-t}e^{-(ixt)}dt$$

$$=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-(ix+1)t}dt \tag{1.23}$$

Apply change of variables:

Let $u=(ix+1)t \Leftrightarrow t=\frac{1}{(ix+1)}u.$ Then $dt=\frac{1}{(ix+1)}du.$

When $t=0,\quad u=0$ and as $t\to\infty,\quad u\to\infty.$ Equation (1.23) then becomes

$$F[f](x)=\frac{1}{(ix+1)\sqrt{2\pi}}\int_{0}^{\infty} e^{-u}du$$

$$F[f](x)=\frac{1}{(ix+1)\sqrt{2\pi}}\int_{0}^{\infty} u^{1-1}e^{-u}du$$

Gamma function (1.1) is then applied to yield

$$F[f](x)=\frac{1}{\sqrt{2\pi}}\left\{\frac{\Gamma(1)}{(ix+1)}\right\}$$

Since $\Gamma(1)=0!=1$, thus

$$F[f](x)=\frac{1}{(ix+1)\sqrt{2\pi}}$$

As required.

The next to consider is the ***one-sided growing exponential*** which is defined as

$$f(t)=\begin{cases}0, & t<0\\ e^{t}, & t\geq 0\end{cases}$$

Proposition 3

The Fourier transform of the one-sided growing exponential is

$$F[f](x)=\frac{1}{(ix-1)\sqrt{2\pi}} \quad . \tag{1.24}$$

Proof

By definition we have

$$F[f](x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-(ixt)}dt$$

$$F[f](x)=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{t}e^{-(ixt)}dt$$

$$=\frac{1}{\sqrt{2\pi}}\left\{\int_{-\infty}^{0} e^{-(ix-1)t}dt\right\} \tag{1.25}$$

Apply change of variables by letting $u=(ix-1)t \Leftrightarrow t=\frac{1}{(ix-1)}u.$ Then $dt=\frac{1}{(ix-1)}du.$

When $t=0, \quad u=0$ and when $t\to\infty, \quad u\to\infty.$

Thus (1.25) becomes

$$F[f](x)=\frac{1}{(ix-1)\sqrt{2\pi}}\int_{0}^{\infty} e^{-u}du$$

$$=\frac{1}{(ix-1)\sqrt{2\pi}}\int_{0}^{\infty} u^{1-1}e^{-u}du \tag{1.26}$$

Comparing (1.26) to the Gamma function (1.1) yields

$$F[f](x)=\frac{1}{(ix-1)\sqrt{2\pi}}\Gamma(1)$$

Which simplifies to

$$F[f](x)=\frac{1}{(ix-1)\sqrt{2\pi}}.$$

The next consideration is the ***double-sided exponential*** which is defined as

$$f(t) = e^{-|t|}, \ \forall t.$$

Proposition 4

The Fourier transform of the double-sided exponential is

$$F[f](x) = \frac{1}{1+x^2}\sqrt{\frac{2}{\pi}}.$$

Proof

Write

$$F[f](x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-(ixt)}dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|t|}e^{-(ixt)}dt \qquad (1.27)$$

Note that

$$|t| = \begin{cases} t, & t > 0 \\ -t, & t < 0 \end{cases}$$

Thus (1.27) becomes

$$\begin{aligned} F[f](x) &= \frac{1}{\sqrt{2\pi}}\left\{\int_{-\infty}^{0} e^{t}e^{-(ixt)}dt + \int_{0}^{\infty} e^{-t}e^{-(ixt)}dt\right\} \\ &= \frac{1}{\sqrt{2\pi}}\left\{\int_{-\infty}^{0} e^{(1-ix)t}dt + \int_{0}^{\infty} e^{-(ix+1)t}dt\right\} \\ &= \frac{1}{\sqrt{2\pi}}\left\{-\int_{0}^{-\infty} e^{(1-ix)t}dt + \int_{0}^{\infty} e^{-(ix+1)t}dt\right\} \\ &= \frac{1}{\sqrt{2\pi}}\left\{\int_{0}^{\infty} e^{-(ix+1)t}dt - \int_{0}^{-\infty} e^{(1-ix)t}dt\right\} \\ &= \frac{1}{\sqrt{2\pi}}\left\{\int_{0}^{\infty} e^{-(ix+1)t}dt + \int_{0}^{\infty} e^{-(1-ix)t}dt\right\} \end{aligned} \qquad (1.28)$$

Let $u = (ix+1)t \Leftrightarrow t = \frac{1}{(ix+1)}u.$ Then $dt = \frac{1}{(ix+1)}du.$

Also let $v = (1-ix)t \Leftrightarrow t = \frac{1}{(1-ix)}v.$ Then $dt = \frac{1}{(1-ix)}dv.$

When $t = 0, \quad u = v = 0$ and when $t \to \infty, \quad u = v \to \infty.$

Equation (1.28) then becomes

$$F[f](x) = \frac{1}{\sqrt{2\pi}}\left\{\int_0^\infty e^{-u}du + \int_0^\infty e^{-v}dv\right\}$$

$$F[f](x) = \frac{1}{\sqrt{2\pi}}\left\{\frac{1}{(ix+1)}\int_0^\infty u^{1-1}e^{-u}du + \frac{1}{(1-ix)}\int_0^\infty v^{1-1}e^{-v}dv\right\} \tag{1.29}$$

Comparing (1.29) to the Gamma function (1.1) yields

$$F[f](x) = \frac{1}{\sqrt{2\pi}}\left\{\frac{\Gamma(1)}{(ix+1)} + \frac{\Gamma(1)}{(1-ix)}\right\}$$

$$F[f](x) = \frac{1}{\sqrt{2\pi}}\left\{\frac{1}{(ix+1)} + \frac{1}{(1-ix)}\right\}$$

Which simplifies to

$$F[f](x) = \frac{1}{\sqrt{2\pi}}\left\{\frac{2}{1+x^2}\right\}$$

$$F[f](x) = \frac{1}{1+x^2}\sqrt{\frac{2}{\pi}}.$$

4 Conclusion

The Gamma function has been studied and presented with illustrative examples to demonstrate its usefulness. It was applied in establishing the legitimacy of the probability density function of a standard normal distribution and also used to derive the moment generating function of a standard normal distribution. The Laplace and Fourier transforms of some common functions were obtained with the application of the Gamma function. This makes the Gamma function a powerful tool in solving some mathematical problems.

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Competing Interests

The authors declare that they have no competing interests.

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