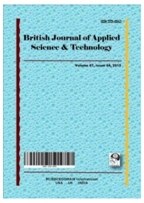

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On the Riesz Integral Representation of Additives Set-Valued Maps (I)

Lakmon A. Kodjovi[1*], Siggini K. Kenny[1], Ayassou Emmanuel[1] and Tcharиè Kokou[1]

[1] *Department of Mathematics, University of Lomé, Togo.*

Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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ABSTRACT

In this paper we generalize the Riesz integral representation for continuous linear maps associated with additive set-valued maps with values in the set of all closed bounded convex non-empty subsets of any Banach space. We deduce the Riesz integral representation results for set-valued maps, for vector-valued maps of Diestel-Uhl and for scalar-valued maps of Dunford-Schwartz.

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**Corresponding author: E-mail: davidlakmon@gmail.com;*

1 INTRODUCTION

The Riesz-Markov-Kakutani representation theorem states that for every positive functional L on the space $C_c(T)$ of continuous compact supported functional on a locally compact Hausdorff space T, there exists a unique Borel regular measure μ on T such that $L(f) = \int f\,d\mu$ for all $f \in C_c(T)$. Riesz's original form [1] was proved in 1909 for the unit interval ($T = [0;1]$). Successive extensions of this result were given, first by Markov in 1938 to some non-compact space (see [2]), by Radon for compact subset of $\mathbb{R}^n$ (see [3]), by Banach in note II of Saks'book [4] and by Kakutani in 1941 to compact Hausdorff space [5]. Others extensions for locally compact spaces are due to Halmos [6], Hewith [7], Edward [8] and N. Bourbaki [9]. Singer [10], [11], Dinculeanu [12], [13] and Diestel-Uhl [14] gave an integral representation for functional on the space $C(T, E)$ of vector-valued continuous functions. Recently Mehdi Ghasemi has shown the integral representation for continuous functionals defined on the space $C(T)$ of all continuous real-valued functions on T; as an application, he gives short solutions for the full and truncated K-moment problem (see [15]). The set-valued measures which are a natural extension of the classical vector measures have been the subject of many thesis. In the school of Pallu De La Barriere we have the ones of: D. S. Thiam [16], A. Costé [17], K. Siggini [18]. In the school of C. Castaing the one of C. Godet-Thobie [19], and in the school of D. S. Thiam the ones of G. Dia [20], M. Thiam [21], G. B. Ndiaye [22]. Investigations are undertaken for the generalization of results for set-valued measures in particular the Radon-Nikodym theorem for weak set-valued measures [23], [24] and the integral representation for additive strictly continuous set-values maps with regular set-valued measures (see [25]). The work of W. Rupp in T arbitrary non-empty set and T compact allowed to generalize the Riesz integral representation of additive and σ-additive scalar measures to the case of additive and σ-additive set-valued measures (see [26]). Among other things he showed that if T is a non-empty set and $\mathfrak{A}$ the algebra of subsets of T, for all continuous linear maps l defined on the space $\mathcal{B}(T, \mathbb{R})$ of all uniform limits of finite linear combinations of characteristic functions of sets in $\mathfrak{A}$ associated with an additive set-valued map with values in the space $ck(\mathbb{R}^n)$ of convex compact non-empty subsets of $\mathbb{R}^n$, there exists a unique bounded additive set-valued measure M from $\mathfrak{A}$ to the space $ck(\mathbb{R}^n)$ such that $\delta^*\left(.\,|l(f)\right) = \delta^*\left(.\,\middle|\int fM\right)$ and conversely. In this paper we prove this result in the case of any Banach space E. We deduce the Riesz integral representation for additive set-valued maps with values in the space of all closed bounded convex non-empty subsets of E; for vector-valued maps (see [14], theorem 13, p.6) and for scalar-valued maps (see [28], theorem 1, p. 258).

2 PRELIMINARIES

Let E be a Banach space and E' its dual space. We denote by $\|.\|$ the norm on E and E'. If X and Y are subsets of E we shall denote by $X+Y$ (resp. $X-Y$) the family of all elements of the form $x + y$ (resp. $x - y$) with $x \in X$ and $y \in Y$, and by $X\dot{+}Y$ or $adh(X + Y)$ the closure of $X + Y$. The closed convex hull of X is denoted $\overline{co}(X)$. The support function of X is the function $\delta^*(.|X)$ from E' to $]-\infty; +\infty]$ defined by $\delta^*(y|X) = \sup\{y(x);\, x \in X\}$. We denote by $cfb(E)$ the set of all closed bounded convex non-empty subsets of E. Let $cfb(E)$ be endowed with the Hausdorff distance denoted by δ and the operations $\dot{+}$ and the multiplication by positive real numbers. For all $K \in cfb(E)$ and for all $K' \in cfb(E), \delta(K, K') = \sup\{|\delta^*(y|K) - \delta^*(y|K')|; y \in E', \|y\| \le 1\}$. Recall that $(cfb(E), \delta)$ is a complete metric space (see [27], proposition 4.2, p. I-13-). We denote by $C^h(E')$ the space of all continuous real-valued map defined on E' and positively homogeneous ie if $u \in C^h(E')$, then $u(\lambda y) = \lambda u(y)$ for all $y \in E'$ and for all $\lambda \in \mathbb{R}$, $\lambda \ge 0$. We endowed $C^h(E')$ with the norm: $\|u\| = \sup\{|u(y)|;\ y \in E', \|y\| \le 1\}$. Put $C_0 = \{\delta^*(.|B);\ B \in cfb(E)\}$ and put $\widetilde{C_0} = C_0 - C_0$; then $\widetilde{C_0}$ is a subspace of the vector space $C^h(E')$ generated by C_0. Let T be a non-empty set, let $\mathfrak{A}$ be the algebra of all subsets of T and let $B(T, \mathbb{R})$ be the space of all bounded real-valued functions defined on T, endowed with the topology of uniform convergence. We denote by $\mathcal{S}(T, \mathbb{R})$ the subspace of $B(T, \mathbb{R})$ consisting of simple functions (i.e. of the form $\sum \alpha_i 1_{A_i}$ where $\alpha_i \in \mathbb{R}$, $A_i \in \mathfrak{A}$, $\{A_1, A_2, ..., A_n\}$ a partition of T and 1_{A_i} the characteristic function of A_i.)

We denote by $\mathcal{B}(T,\mathbb{R})$ the closure in $B(T,\mathbb{R})$ of $\mathcal{S}(T,\mathbb{R})$; $\mathcal{S}_+(T,\mathbb{R})$ (resp. $\mathcal{B}_+(T,\mathbb{R})$) the subspace of $\mathcal{S}(T,\mathbb{R})$ (resp. $\mathcal{B}(T,\mathbb{R})$) consisting of positive functions. Let $\mathcal{B}(T,\mathbb{R})$ be endowed with the induced topology.

Note that if $\mathfrak{A}$ is the Borel σ-algebra, then $\mathcal{B}(T,\mathbb{R})$ is the space of all bounded measurable real-valued functions.

Let M be a set-valued map from $\mathfrak{A}$ to $cfb(E)$. M is called an additive set-valued measure if $M(\emptyset) = \{0\}$ and $M(A \cup B) = M(A)\dot{+}M(B)$ for all disjoint sets A, B in $\mathfrak{A}$. The set-valued measure M is said to be bounded if $\bigcup\{M(A),\ A \in \mathfrak{A}\}$ is a bounded subset of E. The semivariation of M is the map $\|M\|(.)$ from $\mathfrak{A}$ to $[0;+\infty]$ defined by $\|M\|(A) = \sup\{|\delta(y|M(.))|(A);\ y \in E', \|y\| \leq 1\}$ where $|\delta(y|M(.))|(A)$ denotes the total variation of the scalar measure $\delta^*(y|M(.))$ on A defined by $|\delta(y|M(.))|(A) = \sup\sum_i |\delta^*(y|M(A_i))|$; the supremum is taken over all finite partitions (A_i) of $A,\ A_i \in \mathfrak{A}$.

If $\|M\|(T) < +\infty$, then M will be called a set-valued measure of finite semivariation. We denote by $\mathcal{M}(\mathfrak{A}, cfb(E))$ the space of all bounded set-valued measures defined on $\mathfrak{A}$ with values in $cfb(E)$. Let m be a vector measure from $\mathfrak{A}$ to E. We say that m is a bounded additive vector measure if its verifies the similar conditions of bounded additive set-valued measures. We denote by $\|m\|$ the semivariation of m defined by $\|m\|(A) = \sup\{|y \circ m|(A);\ y \in E', \|y\| \leq 1\}$ where $|y \circ m|(A)$ denotes the total variation of the scalar measure $y \circ m$ on A defined by $|y \circ m|(A) = \sup\sum_i |y(m(A_i))|$ for all $A \in \mathfrak{A}$; the supremum is taken over all finite partitions (A_i) of $A,\ A_i \in \mathfrak{A}$.

Let $L : \mathcal{B}_+(T,\mathbb{R}) \to cfb(E)$ be a set-valued map. We say that L is an additive (resp. positively homogeneous) if for all $f, g \in \mathcal{B}_+(T,\mathbb{R})$ (resp. for all $\lambda \geq 0$), $L(f + g) = L(f)\dot{+}L(g)$ (resp. $L(\lambda f) = \lambda L(f)$). We denote by $\mathcal{L}(\mathcal{B}(T,\mathbb{R}), C^h(E'))$ the space of all linear continuous maps defined on $\mathcal{B}(T,\mathbb{R})$ with values in $C^h(E')$. If $l \in \mathcal{L}(\mathcal{B}(T,\mathbb{R}), C^h(E'))$; we put $\|l\| = \sup\{\|l(f)\|;\ f \in \mathcal{B}_+(T,\mathbb{R}), \|f\| \leq 1\}$ where $\|f\| = \sup\{|f(t);\ t \in T|\}$. For a numerical function f defined on T, we set $f^+ = \sup(f, 0)$, and $f^- = \sup(-f, 0)$.

3 MAIN RESULTS

Definition 3.1. Let $l \in \mathcal{L}(\mathcal{B}(T,\mathbb{R}), C^h(E'))$ and let $L : \mathcal{B}_+(T,\mathbb{R}) \to cfb(E)$ be an additive, positively homogeneous and continuous set-valued map. We say that l is associated with L if $l(f) = \delta^*(.|L(f))$ for all $f \in \mathcal{B}_+(T,\mathbb{R})$. Then $l(f) = \delta^*(.|L(f^+)) - \delta^*(.|L(f^-)) \in \widetilde{C_0}$ for all $f \in \mathcal{B}(T,\mathbb{R})$.

Lemma 3.1. *Let $M : \mathfrak{A} \to cfb(E)$ be an additive set-valued measure. Then M is bounded if and only if it is finite semivariation.*

Proof. The set-valued measure M is bounded if there exists a real number $c > 0$ such that $\sup_{A\in\mathfrak{A}} \sup_{\|y\|\leq 1} |\delta^*(y|M(A))| \leq c$. We have:

$$\sup_{A\in\mathfrak{A}} \sup_{\|y\|\leq 1} |\delta^*(y|M(A))| \leq \sup_{\|y\|\leq 1} |\delta^*(y|M(.))|(T) = \|M\|(T).$$

On the other hand, by the lemma 5 ([28], p. 97) one has

$$|\delta^*(y|M(.))|(T) \leq 2 \sup_{A\in\mathfrak{A}} |\delta^*(y|M(A))| \text{ for all } y \in E'.$$

Then $\sup_{\|y\|\leq 1} |\delta^*(y|M(.))|(T) \leq 2 \sup_{A\in\mathfrak{A}} \sup_{\|y\|\leq 1} |\delta^*(y|M(A))|$. Therefore

$$\sup_{A\in\mathfrak{A}} \sup_{\|y\|\leq 1} |\delta^*(y|M(A))| \leq \|M\|(T) \leq 2 \sup_{A\in\mathfrak{A}} \sup_{\|y\|\leq 1} |\delta^*(y|M(A))|.$$

□

Lemma 3.2. *Let C_0 be the set $\{\delta^*(.|B); B \in cfb(E)\}$ and let $l : \mathcal{B}(T,\mathbb{R}) \to C^h(E')$ be a continuous linear map. Then l is associated with an additive, positively homogeneous and continuous set-valued map if and only if $l(f) \in C_0$ for all $f \in \mathcal{B}_+(T,\mathbb{R})$.*

Proof. The necessary condition is obvious. Now assume that $l(f) \in C_0$ for all $f \in \mathcal{B}_+(T,\mathbb{R})$. Let consider the map $j : cfb(E) \to C_0(B \mapsto \delta^*(.|B))$; then j is an isomorphism, more a homeomorphism (see [29], Theorem 8, p.185). Let l' be the restriction of l to $\mathcal{B}_+(T,\mathbb{R})$. Put $L = j^{-1} \circ l'$; then L is additive, positively homogeneous and continuous. Therefore for all $f \in \mathcal{B}_+(T,\mathbb{R}), l(f) = \delta^*(.|L(f)) \in C_0$. □

Let $M : \mathfrak{A} \to cfb(E)$ be a bounded additive set-valued measure.

For all $h \in \mathcal{S}_+(T,\mathbb{R})$ such that $h = \sum a_i 1_{B_i}$ and for all $A \in \mathfrak{A}$, the integral $\int_A hM$ of h with respect to M is defined by:
$\int_A hM = adh\,(a_1 M(A \cap B_1) + a_2 M(A \cap B_2) + ... + a_n M(A \cap B_n))$. This integral is uniquely defined. Moreover for all $y \in E', \delta^*\left(y \left| \int_A hM\right.\right) = \int_A h\delta^*(y|M(.))$. The map: $h \mapsto \int_A hM$ from $\mathcal{S}_+(T,\mathbb{R})$ to $cfb(E)$ is uniformly continuous. Indeed, for all $f, g \in \mathcal{S}_+(T,\mathbb{R})$, one has:

$$\begin{aligned}
\delta\left(\int_A fM, \int_A gM\right) &= \sup_{\|y\|\le 1} \left|\int_A (f-g)\delta^*(y\,|M(.))\right| \\
&\le \sup_{\|y\|\le 1} \|f-g\| |\delta^*(y|M(A))| \\
&\le \|f-g\| \|M\|(T) < +\infty.
\end{aligned}$$

Since $\mathcal{S}_+(T,\mathbb{R})$ is dense on $\mathcal{B}_+(T,\mathbb{R})$ and $cfb(E)$ is a complete metric space, then it has a unique extension to $\mathcal{B}_+(T,\mathbb{R})$: let $f \in \mathcal{B}_+(T,\mathbb{R})$ and let (h_n) be a sequence in $\mathcal{S}_+(T,\mathbb{R})$ converging uniformly to f on T; therefore the integral $\int_A fM$ of f is uniquely defined by $\int_A fM = \lim_{n\to+\infty} \int_A h_n M$. Moreover $\delta^*\left(y \left| \int_A fM\right.\right) = \int_A f\delta^*(y|M(.))$ for all $y \in E'$, $A \in \mathfrak{A}$ and for all $f \in \mathcal{B}_+(T,\mathbb{R})$. The map: $\mathcal{B}_+(T,\mathbb{R}) \to cfb(E)(f \mapsto \int fM)$ is additive, positively homogeneous and uniformly continuous.

If m is a vector measure defined on $\mathfrak{A}$, the integral will be defined in the same manner.

Denotes $\mathcal{L}_0(\mathcal{B}(T,\mathbb{R}), C^h(E'))$ the subspace of $\mathcal{L}(\mathcal{B}(T,\mathbb{R}), C^h(E'))$ consisting of functions that verify the condition $l(f) \in C_0$ for all $f \in \mathcal{B}_+(T,\mathbb{R})$.

Theorem 3.3. *Let $\mathcal{M}(\mathfrak{A}, cfb(E))$ be the space of all bounded additive set-valued measures from $\mathfrak{A}$ to $cfb(E)$. Let $l \in \mathcal{L}_0(\mathcal{B}(T,\mathbb{R}), C^h(E'))$. Then there exists a unique set-valued measure $M \in \mathcal{M}(\mathfrak{A}, cfb(E))$ such that $l(f) = \delta^*\left(. \left| \int fM\right.\right)$ for all $f \in \mathcal{B}_+(T,\mathbb{R})$.*

Conversely for all $M \in \mathcal{M}(\mathfrak{A}, cfb(E))$, the mapping: $f \mapsto \delta^*\left(. \left| \int f^+M\right.\right) - \delta^*\left(. \left| \int f^-M\right.\right)$ from $\mathcal{B}(T,\mathbb{R})$ to $C^h(E')$ is an element of $\mathcal{L}_0(\mathcal{B}(T,\mathbb{R}), C^h(E'))$.
Moreover $\|l\| = \|M\|(T)$.

Proof. Let $l \in \mathcal{L}_0(\mathcal{B}(T,\mathbb{R}), C^h(E'))$. Let us prove the uniqueness of the set-valued measure M. Assume that there exists two set-valued measures $M, M' \in \mathcal{M}(\mathfrak{A}, cfb(E))$ such that

$\delta^*\left(. \left| \int fM\right.\right) = l(f) = \delta^*\left(. \left| \int fM'\right.\right)$ for all $f \in \mathcal{B}_+(T,\mathbb{R})$. Then for all $A \in \mathfrak{A}$

$\delta^*\left(. \left| \int 1_A M\right.\right) = \delta^*\left(. \left| \int 1_A M\right.\right) = l(1_A) = \delta^*\left(. \left| \int 1_A M'\right.\right)$(ie $\delta^*(.|M(A)) = \delta^*(.|M'(A))$. Hence $M(A) = M'(A)$ for all $A \in \mathfrak{A}$. Since $l \in \mathcal{L}_0(\mathcal{B}(T,\mathbb{R}), C^h(E'))$ then l is associated with an additive, positively homogeneous and continuous set-valued map L from $\mathcal{B}_+(T,\mathbb{R})$ to $cfb(E)$. Let $M : \mathfrak{A} \to cfb(E)$ be the set-valued map defined by $M(A) = L(1_A)$ for all $A \in \mathfrak{A}$. Then M is additive. It follows from the continuity of L that M is bounded. Moreover $\int hM = L(h)$ for all $h \in \mathcal{S}_+(T,\mathbb{R})$. Let $f \in \mathcal{B}_+(T,\mathbb{R})$ and let (h_n) be a sequence in $\mathcal{S}_+(T,\mathbb{R})$ converging uniformly to f on T. It follows from the definition of the integral $\int fM$ of f with respect to M and the continuity of L that $L(f) = \lim_{n\to+\infty} L(h_n) = \lim_{n\to+\infty} \int h_n M = \int fM$. Hence $l(f) = \delta^*\left(.\,\middle|\int fM\right)$ for all $f \in \mathcal{B}_+(T,\mathbb{R})$.

Conversely let $M \in \mathcal{M}(\mathfrak{A}, cfb(E))$. Then the map $\theta : \mathcal{B}_+(T,\mathbb{R}) \to C^h(E')$ defined by $\theta(f) = \delta^*\left(.\,\middle|\int f^+M\right) - \delta^*\left(.\,\middle|\int f^-M\right)$ verifies the condition $\theta(f) \in C_0$ for all $f \in \mathcal{B}_+(T,\mathbb{R})$. Let j be the isomorphism from $cfb(E)$ to C_0 defined by $j(B) = \delta^*(.|B)$ and let L be the set-valued map from $\mathcal{B}_+(T,\mathbb{R})$ to $cfb(E)$ defined by $L(f) = \int fM$ for all $f \in \mathcal{B}_+(T,\mathbb{R})$. Then j and L are continuous; therefore $\theta = j \circ L$ is continuous on $\mathcal{B}_+(T,\mathbb{R})$ and then on $\mathcal{B}(T,\mathbb{R})$.

Let us prove now that $\|l\| = \|M\|(T)$. On one hand, for all $y \in E'$

$$\begin{aligned}\|l\| &= \sup_{\|f\|\le 1} \|l(f)\| \\ &= \sup_{|y|\le 1}\sup_{\|f\|\le 1} \left|\delta^*\left(y\middle|\int f^+M\right) - \delta^*\left(y\middle|\int f^-M\right)\right| \\ &= \sup_{|y|\le 1}\sup_{\|f\|\le 1} \left|\int f^+\delta^*(y|M(.)) - \int f^-\delta^*(y|M(.))\right| \\ &= \sup_{|y|\le 1}\sup_{\|f\|\le 1} \left|\int f\delta^*(y|M(.))\right|.\end{aligned}$$

On the other hand $\|M\|(T) = \sup_{\|y\|\le 1} |\delta^*(y|M(.))|(T)$. Then it suffices to prove the equality

$$\sup_{\|f\|\le 1}\left|\int f\delta^*(y|M(.))\right| = |\delta^*(y|M(.))|(T)$$

which is classic. □

Corollary 3.4. *Let L be an additive, positively homogeneous and continuous set-valued map from $\mathcal{B}_+(T,\mathbb{R})$ to $cfb(E)$. Then there is a unique bounded additive set-valued measure M from $\mathfrak{A}$ to $cfb(E)$ such that $L(f) = \int fM$ for all $f \in \mathcal{B}_+(T,\mathbb{R})$.*

Conversely for all bounded additive set-valued measure $M : \mathfrak{A} \to cfb(E)$, the map: $f \mapsto \int fM$ from $\mathcal{B}_+(T,\mathbb{R})$ to $cfb(E)$ is an additive, positively homogeneous and continuous set-valued map.

Proof. The second part follows from the definition of the integral with respect to M.

Let $L : \mathcal{B}_+(T,\mathbb{R}) \to cfb(E)$ be an additive, positively homogeneous and continuous set-valued map and let $j : cfb(E) \to C_0(B \mapsto j(B) = \delta^*(.|B))$. We denote by l the unique extension of $j \circ L$ to $\mathcal{B}(T,\mathbb{R})$: for all $f \in \mathcal{B}(T,\mathbb{R})$ $l(f) = j \circ L(f^+) - j \circ L(f^-) = \delta^*(.|L(f^+)) - \delta^*(.|L(f^-))$. We have $l(f) = \delta^*(.|L(f)) \in C_0$ for all $f \in \mathcal{B}_+(T,\mathbb{R})$; then there exists a unique bounded additive set-valued measure M from $\mathfrak{A}$ to $cfb(E)$ such that $l(f) = \delta^*\left(.\,\middle|\int fM\right)$ for all $f \in \mathcal{B}_+(T,\mathbb{R})$. Hence $L(f) = \int fM$ for all $f \in \mathcal{B}_+(T,\mathbb{R})$. □

The following corollary is partly known (see [14], theorem 13, p.6)

Corollary 3.5. *Let $\mathcal{L}(\mathcal{B}(T,\mathbb{R}), E)$ be the space of all continuous linear maps from $\mathcal{B}(T,\mathbb{R})$ to E and let $\mathcal{M}(\mathfrak{A}, E)$ be the space of all bounded additive vector measures from $\mathfrak{A}$ to E.*

Let $l \in \mathcal{L}(\mathcal{B}(T,\mathbb{R}), E)$. Then there exists a unique vector measure $m \in \mathcal{M}(\mathfrak{A}, E)$ such that $l(f) = \int fm$ for all $f \in \mathcal{B}(T,\mathbb{R})$.

Conversely, given a vector measure $m \in \mathcal{M}(\mathfrak{A}, E)$, the mapping $f \mapsto \int fm$ from $\mathcal{B}(T,\mathbb{R})$ to E is an element of $\mathcal{L}(\mathcal{B}(T,\mathbb{R}), E)$. Moreover $\|l\| = \|m\|(T)$.

Proof. Put $\widetilde{E_0} = \{\{x\};\ x \in E\}$. Then $\widetilde{E_0}$ is a closed subspace of $cfb(E)$. Let j_1 be the map from E to $\widetilde{E_0}$ defined by $j_1(x) = \{x\}$. Then j_1 is an isomorphism more a homeomorphism. let l' be the restriction of $j_1 \circ l$ to $\mathcal{B}_+(T,\mathbb{R})$. Then l' is additive, positively homogeneous and continuous. Therefore by the corollary 3.4 there exists a unique set-valued measure $m' \in \mathcal{M}(\mathfrak{A}, cfb(E))$ such that $l'(f) = \int fm'$ for all $f \in \mathcal{B}_+(T,\mathbb{R})$. It follows from this equality that $m'(A) \in \widetilde{E}_0$ for all $A \in \mathfrak{A}$. Put $m = j_1^{-1} \circ m'$. Then

$m \in \mathcal{M}(\mathfrak{A}, E)$ and verifies $m'(A) = j_1(m(A))$ for all $A \in \mathfrak{A}$. We deduce that $\int fm' = j_1(\int fm)$ for all $f \in \mathcal{B}_+(T, \mathbb{R})$; then $\int fm = j_1^{-1} \circ l'(f) = l(f)$ for all $f \in \mathcal{B}_+(T, \mathbb{R})$ and consequently $l(f) = \int fm$ for all $f \in \mathcal{B}(T, \mathbb{R})$.
The second part of corollary is proved as in the corollary 3.4. The equality $\|l\| = \|m\|(T)$ is a particular case of the theorem 3.3.

□

By putting $E = \mathbb{R}$, we have the following corollary:

Corollary 3.6. *([28], theorem 1, p. 258)*
Let $\mathcal{M}(\mathfrak{A}, \mathbb{R})$ be the space of all bounded additive real-valued measures defined on $\mathfrak{A}$. Let l be a continuous linear functional defined on $\mathcal{B}(T, \mathbb{R})$. Then there exists a unique measure $\mu \in \mathcal{M}(\mathfrak{A}, \mathbb{R})$ such that $l(f) = \int f d\mu$ for all $f \in \mathcal{B}(T, \mathbb{R})$.
Conversely, for all measure $\mu \in \mathcal{M}(\mathfrak{A}, \mathbb{R})$, the mapping: $f \mapsto \int f d\mu$ is a continuous linear functional defined on $\mathcal{B}(T, \mathbb{R})$.
Moreover $\|l\| = |\mu|(T)$.

4 CONCLUSION

We investigate the first part of the Riesz integral representation for continuous linear maps associated with additive set-valued maps with values in the set of all closed bounded convex non-empty subsets of any Banach space, which allows the construction of bounded additive set-valued measures. In particular the integral representation is given for additive, positively homogeneous and continuous set-valued maps, and an alternative proofs are given for the integral representation results for vector-valued maps of Diestel-Uhl and for real-valued maps of Dunford-Schwartz.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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