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A Numerical Integrator for Oscillatory Problems

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Authors' contributions

This work was carried out in collaboration among all authors. Author YDJ proposed the method, derived the method, and wrote the first draft of the manuscript. Authors YBA, AAIM, AA and IM handled the stability analysis of the proposed method and numerical experiment aspect. All authors read and approved the final manuscript.

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Abstract

Presented here is a numerical integrator, with sixth order of convergence, for solving oscillatory problems. Dispersion and dissipation errors are taken into account in the course of deriving the method. As a result, the method possesses dissipation of order infinity and dispersive of order six. Validity and effectiveness of the method are tested on a number of test problems. Results obtained show that the new method is better than its equals in the scientific literature.

Keywords: Dispersion; dissipation; oscillatory problems; differential equations; numerical experiment.

Mathematics classification: 65L05.

1 Introduction

The problem of interest in this paper is an initial value problem (IVP) of the form:

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$$y'' = f(x, y), y(x_0) = y_0, y'(x_0) = y'_0. \tag{1}$$

The importance of this problem cannot be over emphasized in science and engineering, as it arises in many areas including quantum mechanics, semi-discretizations of wave equation, astrophysics, electronics, celestial mechanics, quantum chemistry, molecular dynamics, and so on. For example, Schrödinger equation and many-body problem are oscillatory problems [1]. To validate models that exist in the form of (1), their solutions are of paramount importance. The best form of solution of (1) is exact or analytical solution. Research has shown that only a few differential equations (DEs) can be solved exactly or analytically. Hence, numerical techniques for obtaining approximate solution of the equations become the best option. A number of numerical methods for solving (1) have been proposed in the literature [2–5,6–24,1,25,26]. Some are direct while some are indirect methods. The direct methods require no transformation of (1) into system of first order equations, while indirect methods do.

The method proposed in this paper has the following general form:

$$\begin{aligned} y_{n+1} &= 2y_n - y_{n-1} + h^2 \sum_{i=1}^{m} b_i f(x_n + c_i h, Y_i), i = 1...m, \\ Y_i &= (c_i + 1)y_n - c_i y_{n-1} + h^2 \sum_{j=1}^{i} a_{ij} f(x_n + c_j h, Y_j). \end{aligned} \tag{2}$$

Where y_n and y_{n-1} are approximations for $y(x_n)$ and $y(x_{n-1})$, respectively. The parameters a_{ij}, b_i, c_i are real numbers. Let $\mathbf{A}=[a_{ij}]$ be an $m \times m$ matrix and **b** and **c** given by $\mathbf{b}=[b_1, b_2, ..., b_m]^{\mathrm{T}}$ and $\mathbf{c}=[\mathrm{c}_1, \mathrm{c}_2, ..., c_m]^{\mathrm{T}}$ be *m*-dimensional vectors, then we can summarize the coefficients of (2) in a Butcher-like tableau as follows:

$$\begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \mathbf{b}^{\mathrm{T}} \end{array}$$

The method (1) is called hybrid method [9], because it is non linear and not self starting. Franco [9] derived the coefficients of an explicit class of (2) up to algebraic order six with less computation cost by using the algebraic order conditions of the method developed in [27]. To improve its properties for better performance, Ahmad et al. [28] proposed semi-implicit hybrid methods up to algebraic order five. In the work of Jikantoro et al. [1] it is pointed out that there is a huge advantage of improved accuracy and efficiency as the order of this method increases. Hence, we propose a semi-implicit hybrid method with higher algebraic order.

The remaining part of the paper is organized as follows: dispersion and dissipation analysis is presented in Section 2. The proposed method is fully derived in Section 3. Stability property of the method is analyzed in Section 4. Results of numerical experiment are presented in Section 5. Section 6 is conclusion.

2 Dispersion and Dissipation Analysis

To analyze errors due to dispersion and dissipation or phase lag and amplification factor, a simple homogeneous test equation below is considered.

$$y'' = -\lambda^2 y, \lambda \succ 0 \in R. \tag{3}$$

Apply eqn. (2) eqn. (3), we get the following in vector form:

$$\begin{aligned}&\mathbf{Y}=(\mathbf{c}+\mathbf{e})y_n-\mathbf{c}y_{n-1}-v^2\mathbf{A}\mathbf{Y},\\&y_{n+1}=2y_n-y_{n-1}-v^2\mathbf{b}^{\mathrm{T}}\mathbf{Y}.\end{aligned} \tag{4}$$

Where $v=\lambda h$, $\mathbf{e}=[1,1,...,1]^{\mathrm{T}}$, vectors **b**, **c** and matrix **A** are given in Section 1. From (4) we get

$$\mathbf{Y}=\left(v^2\mathbf{A}+\mathbf{I}\right)^{-1}(\mathbf{c}+\mathbf{e})y_n-\left(v^2\mathbf{A}+\mathbf{I}\right)^{-1}\mathbf{c}y_{n-1}, \tag{5}$$

which when substituted in second part of eqn. (4) we get

$$y_{n+1}-T_1(v^2)y_n+T_2(v^2)y_{n-1}=0. \tag{6}$$

Where

$$T_1(v^2)=\frac{2+\sum_{i=1}^{m}U_iv^{2i}}{\prod_{i=1}^{m}\left(1+v^2a_{ii}\right)},T_2(v^2)=\frac{1+\sum_{i=1}^{m}V_iv^{2i}}{\prod_{i=1}^{m}\left(1+v^2a_{ii}\right)}. \tag{7}$$

V_i and U_i are functions of the coefficients of the method.

It has been shown in [23] that the solution of (6) can be written as

$$y_n=2\,|\,\mathbb{C}\,||\,\rho\,|^n\cos(\omega+n\phi), \tag{8}$$

where $\mathbb{C},\omega$ are determined by coefficients of the method and ρ,ϕ are amplification factor and phase, respectively. The solution of the test problem (3) is given by

$$y(x_n)=2\,|\,\delta\,|\cos(\varphi+nv), \tag{9}$$

where δ and φ are real constants determined by initial conditions and n is the number of terms. The definition formulated in [23] adopted by [9,1] follows.

Definition 1. *From eqns. (8) and (9), the quantity* $R(v)=v-\phi$ *is referred to dispersion error or phase lag of the method. The method is said to have dispersion error of order q if* $R(v)=O\left(v^{q+1}\right)$. *Furthermore, the quantity* $S(v)=1-|\,\rho\,|$ *is referred to dissipation error of the method. And the method is said to be dissipative of order r if* $S(v)=O\left(v^{r+1}\right)$ *[9].*

From **Definition 1**, it follows that

$$R(v)=v-\cos^{-1}\left(\frac{T_1\left(v^2\right)}{2\sqrt{T_2\left(v^2\right)}}\right),S(v)=1-\sqrt{T_2\left(v^2\right)}. \tag{10}$$

Taking the Taylor expansions of (10) when $m = 4$ yields the order conditions of dispersion and dissipation errors as follows:

Table 1. Dispersion and dissipation conditions

Order	Condition
	Dispersion
4	$V_1 - U_1 = 1, U_2 + \frac{1}{2}V_1 - V_2 + \frac{1}{4}V_1^2 = \frac{1}{12}$
6	$\frac{1}{24}V_1 - U_3 - \frac{1}{2}V_2 + \frac{1}{8}V_1^2 + V_3 - \frac{1}{2}V_1V_2 + \frac{1}{8}V_1^3 = \frac{1}{360}$
8	$V_1V_3 + \frac{1}{2}V_2^{\,2} - \frac{3}{4}V_1^2V_2 + \frac{5}{32}V_1^4 + \frac{1}{360}V_1 + \frac{1}{48}V_1^2 + V_3 - \frac{1}{2}V_1V_2 + \frac{1}{8}V_1^3 = \frac{1}{10080}$
	Dissipation
5	$V_1 = 0, V_2 - \frac{1}{4}V_1^2 = 0$
7	$\frac{1}{2}V_1V_2 + \frac{1}{8}V_1^3 - V_3 = 0$
9	$V_1V_3 + \frac{1}{2}V_2^{\,2} + \frac{5}{32}V_1^4 - \frac{3}{4}V_1^2V_2 = 0$

3 Stability Analysis

Like dispersion and dissipation errors, to analyze stability of (2), the method is applied on the test eqn. (3). Refer to Section 2 for this task. The stability polynomial is obtained by re-writing eqn. (6) as follows:

$$\chi^2 - T_1(v^2)\chi + T_2(v^2) = 0. \tag{11}$$

The solution represented by eqn. (6) is expected to be periodic, provided that its coefficients satisfy the conditions below:

$$T_2\left(v^2\right) \equiv 1, \left|T_1\left(v^2\right)\right| \prec 2, \text{ for all } v^2 \in \left(0, v_p^2\right),$$

where $\left(0, v_p^2\right)$ refers to periodicity interval of the method. When this happens, the method is called a zero-dissipative, because $S(v) = 0$, which means the method has dissipation of order infinity. But if $S(v) \neq 0$, then the method is absolutely stable, provided the following conditions are satisfied:

$$T_2\left(v^2\right) \prec 1, \left|T_1\left(v^2\right)\right| \prec 1 + T_2\left(v^2\right), \text{ for all } v^2 \in \left(0, v_s^2\right),$$

and $\left(0, v_s^2\right)$ is now referred to as absolute stability interval of the method. Detail of this analysis can be found in [27,9,1].

4 Derivation of the Method

In this section, the proposed method is derived. To derive the method, algebraic order conditions of convergent is required.

4.1 Order condition of hybrid method

The algebraic order condition of a numerical method, in general, is a relationship between coefficients of the method that causes the successive terms in the Taylor series expansion of the method to varnish, see Coleman [9]. The algebraic order conditions of the proposed method in this paper have been derived in [9]. They are as follows:

Order 1: $\sum b_i = 1,$

Order 2: $\sum b_i c_i = 0,$

Order 3: $\sum b_i c_i^2 = \frac{1}{6}, \sum b_i a_{ij} = \frac{1}{12},$

Order 4: $\sum b_i c_i^3 = 0, \sum b_i c_i a_{ij} = \frac{1}{12}, \sum b_i a_{ij} c_j = 0,$

Order 5: $\sum b_i c_i^4 = \frac{1}{15}, \sum b_i c_i^2 a_{ij} = \frac{1}{30}, \sum b_i c_i a_{ij} c_j = -\frac{1}{60}, \sum b_i a_{ij} a_{ik} = \frac{7}{120}, \sum b_i a_{ij} c_j^2 = \frac{1}{180},$

$$\sum b_i a_i j a_{jk} c_k = \frac{1}{360},$$

Order 6: $\sum b_i c_i^5 = 0, \sum b_i c_i^3 a_{ij} = \frac{1}{30}, \sum b_i c_i^2 a_{ij} c_j = 0, \sum b_i c_i a_{ij} a_{ik} = \frac{1}{30}, \sum b_i c_i a_{ij} c_j^2 = \frac{1}{72},$

$$\sum b_i c_i a_{ij} a_{jk} = -\frac{1}{720}, \sum b_i a_{ij} a_{jk} c_k = -\frac{1}{120}, \sum b_i a_{ij} c_j^3 = 0, \sum b_i a_{ij} c_j a_{ik} = \frac{1}{360},$$
$$\sum b_i a_{ij} c_{jk} c_k = 0.$$

The order conditions have the following simplifying assumption:

$$\sum_{j=1}^{m} a_{ij} c_j^{\lambda} = \frac{c_i^{\lambda+2} + (-1)^{\lambda} c_i}{(1+\lambda)(2+\lambda)}, \lambda \geq 0. \qquad (12)$$

Deriving sixth order method requires us to solve the all the equations involved in the order conditions up to order six, as stated above. Obviously, there are twenty three equations involved to be solved in thirteen unknown parameters for $m = 4$. This is not possible because the number of equations exceeds the number of unknowns. Suppose we impose eqn. (12) with $\lambda = 0$, then the equations reduce to fifteen, which are still more in number than the unknowns. Next, we impose (12) with $\lambda = 1$, then the equations reduce to thirteen, which are equal in number with the unknowns. We solve the system of thirteen equations for the thirteen unknowns, where a method with unique coefficients is obtained. Below is a summary of the method in a Butcher-like tableau:

Table 2. Coefficients of the proposed method

-1	0			
0	0	0		
$\sqrt{\frac{2}{5}}$	$-\frac{1}{20}\sqrt{\frac{2}{5}}+\frac{1}{30}$	$-\frac{11}{30}\sqrt{\frac{2}{5}}+\frac{13}{60}$	$-\frac{1}{12}\sqrt{\frac{2}{5}}-\frac{1}{20}$	
$-\sqrt{\frac{2}{5}}$	$\frac{1}{20}\sqrt{\frac{2}{5}}-\frac{1}{30}$	$\frac{11}{30}\sqrt{\frac{2}{5}}+\frac{7}{60}$	$\frac{1}{12}\sqrt{\frac{2}{5}}+\frac{1}{12}$	$\frac{1}{30}$
	0	$\frac{7}{12}$	$\frac{5}{24}$	$\frac{5}{24}$

Note that the equations of dissipation conditions up to order nine are contained in the equations of algebraic order conditions of the method. Therefore, the method, which is denoted by ZDSIHM6 (4, 6, ∞), is zero dissipative, dispersive of order six with dispersion constant $\frac{259}{12049477}v^7 + O\left(v^9\right)$ and $(0,\ 4.47)$ as interval of periodicity.

5 Numerical Results

Numerical results of ZDSIHM6 (4, 6, ∞) alongside existing methods when applied to some test problems are presented in this section.

5.1 Test problems

Test problems here mean those oscillatory problems whose solutions are known, so that when the proposed method is applied on them we can compare the approximate solutions with their exact solution to know how good the method can approximate their solutions.

Problem 1: $y''(x) = -64y(x), y(0) = 1, y'(0) = -2, y(x) = -\frac{1}{4}\sin(8x) + \cos(8x), x \in \left[0, 4000\right].$

Problem 2:
$$y''(x) = -v^2 y(x) + \left(-1 + v^2\right)\sin(x), y(0) = 1, y'(0) = 1 + v, y(x) = \sin(vx) + \cos(vx) + \sin(x), v = 10,$$
$$x \in \left[0, 4000\right].$$

Problem 3: $y''(x) = -y(x) + x, y(0) = 1, y'(0) = 2, y(x) = \cos(8x) + \sin(x) + x, x \in \left[0, 4000\right].$

The following acronyms are used in the paper:

- **ZDSIHM6(4,6,∞):** Zero-dissipative sixth order four stage semi-implicit hybrid method derived in this paper.
- **EHM6(5,6,∞):** Existing method obtained in Franco [9].
- **BRKN5(6)FSAL:** Existing code presented in Bettis [3].

Figs. 1-3 give a graphical explanation of the efficiency and accuracy of ZDSIHM6 (4, 6, ∞), together with some of existing methods, measured by plotting the logarithm of maximum error against the logarithm of total function call in the interval *0* to *4000* for each of the problems. It can be seen on the figures that the curve of the proposed method lays below every other curve all through. The interpretation of this is that the method approximated the solutions of the problems with lesser error and cost.

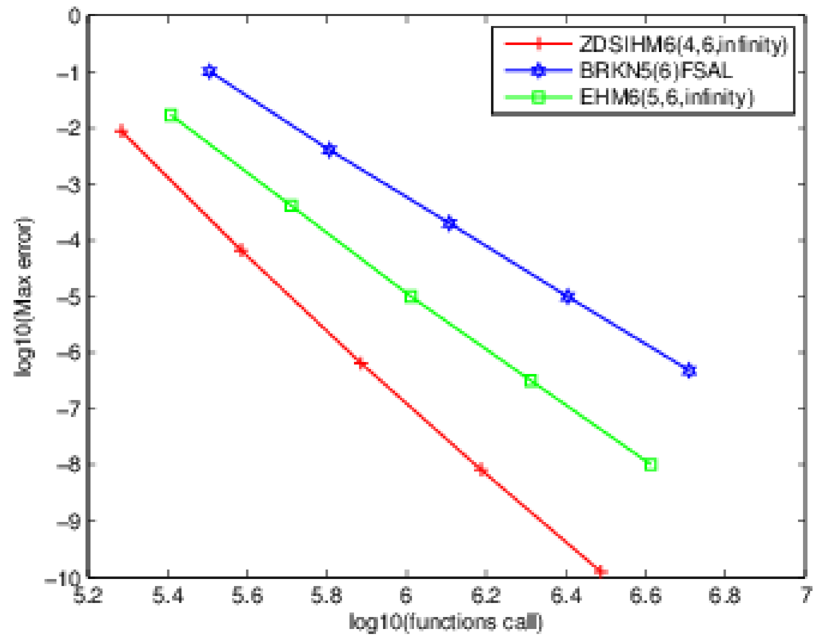

Fig. 1. Efficiency curve of ZDSIHM6 (4, 6, ∞) for problem 1

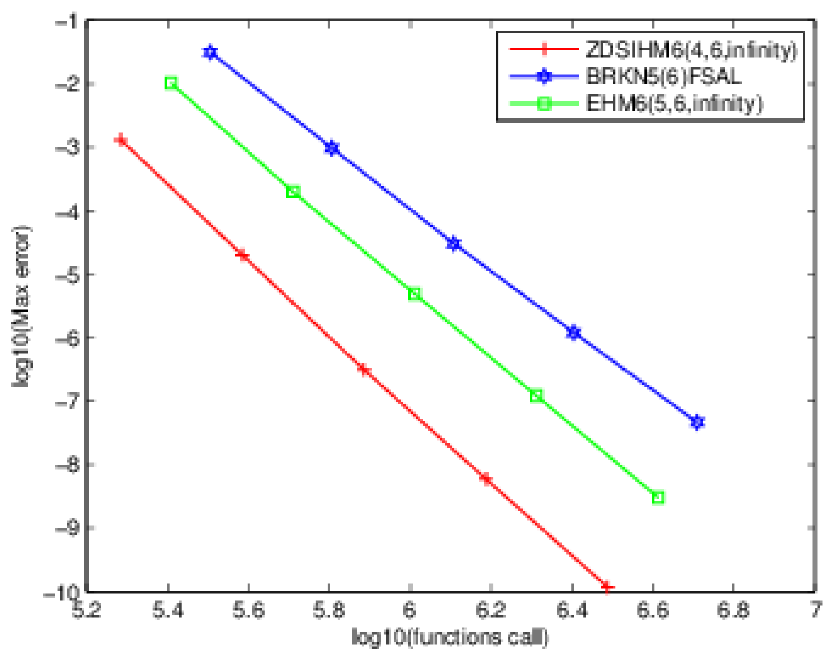

Fig. 2. Efficiency curve of ZDSIHM6 (4, 6, ∞) for problem 2

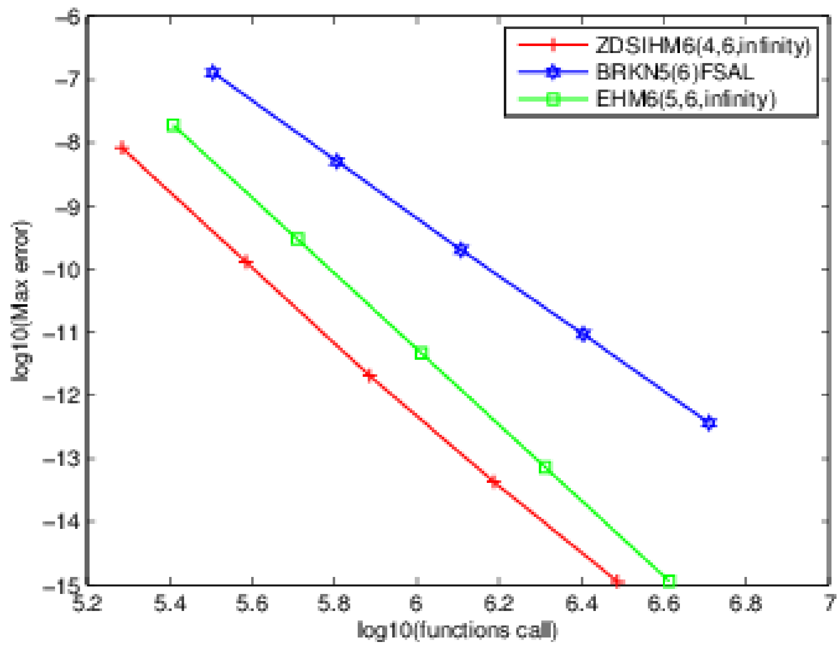

Fig. 3. Efficiency curve of ZDSIHM6 (4, 6, ∞) for problem 3

6 Conclusion

A numerical integrator of hybrid method class is proposed and presented in this paper. The method has algebraic order six; it is dispersive and dissipative of orders six and infinity, respectively. The method is stable. Numerical results reveal that the method is more accurate and efficient than the existing methods considered in the paper.

Competing Interests

Authors have declared that no competing interests exist.

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