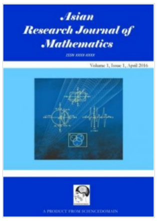

Asian Research Journal of Mathematics

14(1): 1-9, 2019; Article no.ARJOM.49320

ISSN: 2456-477X

On Some Inequalities for the Chaudhry-Zubair Extension of the Gamma Function

Monica Atogpelge Atugba[1] and Kwara Nantomah[1*]

[1] *Department of Mathematics, Faculty of Mathematical Sciences, University for Development Studies, Navrongo Campus, P. O. Box 24, Navrongo, UE/R, Ghana.*

Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

Article Information

DOI: 10.9734/ARJOM/2019/v14i130117

Editor(s):

(1) Dr. Ruben Dario Ortiz Ortiz, Professor, Facultad de Ciencias Exactas y Naturales, Universidad de Cartagena, Colombia.

Reviewers:

(1) Erdal nlyol, Ordu University, Turkey.

(2) Cristescu Gabriela, Aurel Vlaicu University of Arad, Romnia.

(3) F. Mge Sakar, Dicle University, Turkey.

Complete Peer review History: http://www.sdiarticle3.com/review-history/49320

Received: 21 March 2019

Accepted: 27 May 2019

Published: 03 June 2019

Original Research Article

Abstract

By applying the classical Holder's inequality, Young's inequality, Minkowski's inequality and some other analytical tools, we establish some inequalities involving the Chaudhry-Zubair extension of the gamma function. The established results serve as generalizations of some known results in the literature.

Keywords: Chaudhry-Zubair extension; gamma function; log-convex function; Holder's inequality; Young's inequality; Minkowski's inequality; inequality.

2010 Mathematics Subject Classification: 33B15, 26A51, 26D07, 26D15

**Corresponding author: E-mail: knantomah@uds.edu.gh;*

1 Introduction

The classical Euler's gamma function, which is an extension of the factorial notation to non-integer values, is usually defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad x > 0. \tag{1}$$

It was first defined by Leonhard Euler and later studied by several renowned mathematicians. Since its introduction, the gamma has found and continue find useful applications in almost all branches of mathematics. As a result of its importance, it has been studied extensively. Several extensions and generalizations have also been established. For example, see [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11]. Closely associated with the gamma function is the digamma function which is defined as

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad x > 0.$$

In this paper, our focus is on the Chaudhry-Zubair extension of the gamma function which is defined as [2]

$$\Gamma_p(x) = \int_0^\infty t^{x-1} e^{\left(-t-\frac{p}{t}\right)} dt, \quad p > 0, x > 0, \tag{2}$$

where, $\Gamma_p(x)$ tends to the classical gamma function $\Gamma(x)$ when $p = 0$. It satisfies the following identities

$$\Gamma_p(x+1) = x\Gamma_p + p\Gamma_p(x-1),$$
$$\Gamma_p(-x) = p^{-x}\Gamma_p(x).$$

The Chaudhry-Zubair extension of the gamma function has attracted the attention of several researchers and it has be investigated in diverse ways (see [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], and the related references therein).

By differentiating (2) repeatedly, one obtains

$$\Gamma_p^{(n)}(x) = \int_0^\infty (\ln t)^n t^{x-1} e^{\left(-t-\frac{p}{t}\right)} dt, \quad p > 0 \tag{3}$$

where $n \in \mathbb{N}_0$. It is clear that $\Gamma_p^{(n)}(x)$ returns to $\Gamma_p(x)$ when $n = 0$. Here, and for the rest of the paper, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{N} = \{1, 2, 3, \dots\}$.

The aim of this paper is to establish some properties such as log-convexity, monotonicity and inequalities concerning the Chaudhry-Zubair extension of the gamma function. The results established serve as generalizations of some known results in the literature.

2 Preliminaries

The following lemmas are well known in the scientific community. For example, see [24], [25] or [26].

Lemma 2.1 (Holder's Inequality). *Let $r > 1$, $s > 1$ and $\frac{1}{r} + \frac{1}{s} = 1$. If $f(t)$ and $g(t)$ are continuous real-valued functions on $[a, b]$, then inequality*

$$\int_a^b |f(t)g(t)|\, dt \le \left(\int_a^b |f(t)|^r\, dt \right)^{\frac{1}{r}} \left(\int_a^b |g(t)|^s\, dt \right)^{\frac{1}{s}}, \tag{4}$$

holds .

Lemma 2.2 (Young's Inequality). *If $u \geq 0$, $v \geq 0$, $\alpha, \beta \in (0,1)$ such that $\alpha+\beta=1$, then inequality*

$$u^{\alpha} v^{\beta} \leq \alpha u + \beta v, \tag{5}$$

holds .

Lemma 2.3 (Minkowski's Inequality). *Let $u \geq 1$. If $f(t)$ and $g(t)$ are continuous real-valued functions on $[a,b]$, then inequality*

$$\left(\int_a^b |f(t)+g(t)|^u \, dt \right)^{\frac{1}{u}} \leq \left(\int_a^b |f(t)|^u \, dt \right)^{\frac{1}{u}} + \left(\int_a^b |g(t)|^u \, dt \right)^{\frac{1}{u}}, \tag{6}$$

holds .

Definition 2.4. A function $f : I \to (0, \infty)$ is said to be log-convex if $\ln f$ is convex on I. That is if

$$\ln f(\alpha x + \beta y) \leq \alpha \ln f(x) + \beta \ln f(y)$$

or equivalently

$$f(\alpha x + \beta y) \leq (f(x))^{\alpha} (f(y))^{\beta}$$

for each $x, y \in I$ and $\alpha, \beta \in (0,1)$ such that $\alpha + \beta = 1$.

3 Results and Discussion

Theorem 3.1. *The Chaudhry-Zubair extended gamma function satisfies the inequality*

$$\Gamma_p^{(\alpha m + \beta n)}(\alpha x + \beta y) \leq \left[\Gamma_p^{(m)}(x) \right]^{\alpha} \left[\Gamma_p^{(n)}(y) \right]^{\beta}, \tag{7}$$

where, $x > 0$, $y > 0$, $\alpha, \beta \in (0,1)$, $\alpha + \beta = 1$, and $m, n \in \{2s : s \in \mathbb{N}_0\}$.

Proof. Let $x > 0$, $y > 0$, $\alpha, \beta \in (0,1)$, $\alpha + \beta = 1$, and $m, n \in \{2s : s \in \mathbb{N}_0\}$. Then by using (3), we obtain

$$\begin{aligned}
\Gamma_p^{(\alpha m + \beta n)}(\alpha x + \beta y) &= \int_0^{\infty} (\ln t)^{\alpha m + \beta n} t^{(\alpha x + \beta y) - (\alpha + \beta)} e^{\left(-t - \frac{p}{t}\right)(\alpha + \beta)} dt \\
&= \int_0^{\infty} (\ln t)^{m\alpha} (\ln t)^{n\beta} t^{\alpha(x-1)} t^{\beta(y-1)} e^{\alpha(-t-\frac{p}{t})} e^{\beta(-t-\frac{p}{t})} dt \\
&= \int_0^{\infty} (\ln t)^{\alpha m} t^{\alpha(x-1)} e^{\alpha(-t-\frac{p}{t})} (\ln t)^{\beta n} t^{\beta(y-1)} e^{\beta(-t-\frac{p}{t})} dt.
\end{aligned}$$

Then by virtue of the Holder's inequality, we obtain

$$\begin{aligned}
&\int_0^{\infty} (\ln t)^{\alpha m} t^{\alpha(x-1)} e^{\alpha(-t-\frac{p}{t})} (\ln t)^{\beta n} t^{\beta(y-1)} e^{\beta(-t-\frac{p}{t})} dt \\
&\leq \left[\int_0^{\infty} \left[(\ln t)^{\alpha m} t^{\alpha(x-1)} e^{\alpha(-t-\frac{p}{t})} \right]^{\frac{1}{\alpha}} dt \right]^{\alpha} \left[\int_0^{\infty} \left[(\ln t)^{\beta n} t^{\beta(y-1)} e^{\beta\left(-t-\frac{p}{t}\right)} \right]^{\frac{1}{\beta}} dt \right]^{\beta} \\
&= \left[\int_0^{\infty} (\ln t)^m t^{x-1} e^{(-t-\frac{p}{t})} dt \right]^{\alpha} \left[\int_0^{\infty} (\ln t)^n t^{(y-1)} e^{(-t-\frac{p}{t})} dt \right]^{\beta} \\
&= \left[\Gamma_p^{(m)}(x) \right]^{\alpha} \left[\Gamma_p^{(n)}(y) \right]^{\beta}.
\end{aligned}$$

Hence,

$$\Gamma_p^{(\alpha m+\beta n)}(\alpha x+\beta y) \leq \left[\Gamma_p^{(m)}(x)\right]^{\alpha}\left[\Gamma_p^{(n)}(y)\right]^{\beta}.$$

□

Corollary 3.2. *The Chaudhry-Zubair's extended function satisfies the inequality*

$$\Gamma_p^{(n)}(\alpha x+\beta y) \leq \left[\Gamma_p^{(n)}(x)\right]^{\alpha}\left[\Gamma_p^{(n)}(y)\right]^{\beta}, \tag{8}$$

where, $x > 0$, $y > 0$, $\alpha, \beta \in (0,1)$, $\alpha+\beta = 1$ *and* $n \in \{2s : s \in \mathbb{N}_0\}$.

Proof. Let $m = n$ in Theorem 3.1. □

Corollary 3.3. *The inequality*

$$\Gamma_p^{(\frac{m+n}{2})}\left(\frac{x+y}{2}\right) \leq \sqrt{\Gamma_p^{(m)}(x)\Gamma_p^{(n)}(y)}, \tag{9}$$

holds for $x > 0$, $y > 0$ *and* $m, n \in \{2s : s \in \mathbb{N}_0\}$.

Proof. Let $\alpha = \beta = \frac{1}{2}$ in Theorem 3.1. □

Remark 3.4. Corollary 3.3 is a generalization of the results of Mortici as presented in Theorem 2.1 of [27].

Corollary 3.5. *The function,* $\Gamma_p(x)$ *satisfies the inequality*

$$\Gamma_p(\alpha x+\beta y) \leq [\Gamma_p(x)]^{\alpha}[\Gamma_p(y)]^{\beta}, \tag{10}$$

where, $x > 0$, $y > 0$, $\alpha, \beta \in (0,1)$ *and* $\alpha+\beta = 1$.

Proof. Let $m= n = 0$ in Theorem 3.1. □

Remark 3.6. Corollary 3.2 is another way of saying that the Chuahdry-Zubair gamma function is log-convex. This fact has been established in Theorem 4.1 of [28].

Corollary 3.7. *The function,* $\Gamma_p(x)$ *satisfies the inequality*

$$\Gamma_p(x)\Gamma_p''(x) \geq \left[\Gamma_p'(x)\right]^2. \tag{11}$$

In other words,

$$\det\begin{bmatrix} \Gamma_p(x) & \Gamma_p'(x) \\ \Gamma_p'(x) & \Gamma_p''(x) \end{bmatrix} \geq 0. \tag{12}$$

Proof. Since $\Gamma_p(x)$ is log-convex, then $[\ln\Gamma_p(x)]'' \geq 0$ for all $x > 0$. Then

$$[\ln\Gamma_p(x)]'' = \left[\frac{\Gamma_p'(x)}{\Gamma_p(x)}\right]' = \frac{\Gamma_p(x)\Gamma_p''(x) - [\Gamma_p'(x)]^2}{[\Gamma_p(x)]^2} \geq 0.$$

Hence

$$\Gamma_p(x)\Gamma_p''(x) - [\Gamma_p'(x)]^2 \geq 0,$$

which completes the proof. □

Corollary 3.8. *Let* $\psi_p(x) = \frac{\Gamma_p'(x)}{\Gamma_p(x)}, x > 0$ *be the Chaudhry-Zubair extension of the digamma.Then* $\psi_p(x)$ *is increasing.*

Proof. By direct differentiation and by Corollary 3.7, we have,

$$\psi_p'(x) = \left[\frac{\Gamma_p'(x)}{\Gamma_p(x)}\right]' \geq 0,$$

which concludes the proof. □

Theorem 3.9. *The function $\Gamma_p(x)$ satisfies the inequality*

$$\Gamma_p(x+y) \leq \left[\Gamma_p\left(\frac{x}{\alpha}\right)\right]^{\alpha}\left[\Gamma_p\left(\frac{y}{\beta}\right)\right]^{\beta}, \tag{13}$$

for $x > 0$, $y > 0$, $\alpha, \beta \in (0,1)$ and $\alpha + \beta = 1$.

Proof. By definition (2), we have

$$\begin{aligned}\Gamma_p(x+y) &= \int_0^\infty t^{(x+y)-1}e^{(-t-\frac{p}{t})}dt \\ &= \int_0^\infty t^{(x+y)-(\alpha+\beta)}e^{(-t-\frac{p}{t})(\alpha+\beta)}dt \\ &= \int_0^\infty t^{x-\alpha}e^{(-t-\frac{p}{t})\alpha}t^{y-\beta}e^{(-t-\frac{p}{t})\beta}dt,\end{aligned}$$

and by employing the Holders inequality, we obtain

$$\begin{aligned}&\int_0^\infty t^{x-\alpha}e^{(-t-\frac{p}{t})\alpha}t^{y-\beta}e^{(-t-\frac{p}{t})\beta}dt \\ &\leq \left[\int_0^\infty \left[t^{x-\alpha}e^{(-t-\frac{p}{t})\alpha}\right]^{\frac{1}{\alpha}}dt\right]^{\alpha}\left[\int_0^\infty \left[t^{y-\beta}e^{(-t-\frac{p}{t})\beta}\right]^{\frac{1}{\beta}}dt\right]^{\beta} \\ &= \left[\int_0^\infty t^{(\frac{x}{\alpha})-1}e^{(-t-\frac{p}{t})}dt\right]^{\alpha}\left[\int_0^\infty t^{(\frac{y}{\beta})-1}e^{(-t-\frac{p}{t})}dt\right]^{\beta} \\ &= \left[\Gamma_p\left(\frac{x}{\alpha}\right)\right]^{\alpha}\left[\Gamma_p\left(\frac{y}{\beta}\right)\right]^{\beta}.\end{aligned}$$

Hence,

$$\Gamma_p(x+y) \leq \left[\Gamma_p\left(\frac{x}{\alpha}\right)\right]^{\alpha}\left[\Gamma_p\left(\frac{y}{\beta}\right)\right]^{\beta}.$$

□

Corollary 3.10. *The inequality*

$$\Gamma_p(x+y) \leq \alpha\Gamma_p\left(\frac{x}{\alpha}\right) + \beta\Gamma_p\left(\frac{y}{\beta}\right), \tag{14}$$

holds where $x > 0$, $y > 0$, $\alpha, \beta \in (0,1)$ and $\alpha + \beta = 1$.

Proof. From Theorem 3.9 it is obtained that,

$$\Gamma_p(x+y) \leq \left[\Gamma_p\left(\frac{x}{\alpha}\right)\right]^{\alpha}\left[\Gamma_p\left(\frac{y}{\beta}\right)\right]^{\beta}, \tag{15}$$

and then by Young's inequality (5), we have

$$\left[\Gamma_p\left(\frac{x}{\alpha}\right)\right]^{\alpha}\left[\Gamma_p\left(\frac{y}{\beta}\right)\right]^{\beta} \leq \alpha\Gamma_p\left(\frac{x}{\alpha}\right)\beta\Gamma_p\left(\frac{y}{\beta}\right). \tag{16}$$

Now, combining equations (15) and (16) gives (14). □

Theorem 3.11. *For $a \geq 1$, the function*

$$f(x) = \frac{\Gamma_p(ax)}{[\Gamma_p(x)]^a}, \tag{17}$$

is increasing on $(0, \infty)$ and as a result, the inequality

$$\left(\frac{\Gamma_p(y)}{\Gamma_p(x)}\right)^a \leq \frac{\Gamma_p(ay)}{\Gamma_p(ax)} \tag{18}$$

is satisfied for $0 < x < y$.

Proof. Let $g(x) = \ln f(x) = \ln\Gamma_p(ax) - a\ln\Gamma_p(x)$ for $x > 0$. Then

$$\begin{aligned} g(x)' &= a\frac{\Gamma_p'(ax)}{\Gamma_p(ax)} - a\frac{\Gamma_p'(x)}{\Gamma_p(x)}, \\ &= a\left[\psi_p(ax) - \psi_p(x)\right] > 0, \end{aligned}$$

since $\psi_p(x)$ is increasing. Thus, $g(x)$ is increasing. Consequently, $f(x)$ is also increasing. Then for $0 < x < y$, we have $f(x) \leq f(y)$ which gives

$$\frac{\Gamma_p(ax)}{[\Gamma_p(x)]^a} \leq \frac{\Gamma_p(ay)}{[\Gamma_p(y)]^a},$$

and by rearrangement, we obtain (18). □

Theorem 3.12. *The inequality*

$$\left[\Gamma_p^{(m)}(x) + \Gamma_p^{(n)}(y)\right]^{\frac{1}{k}} \leq \left[\Gamma_p^{(m)}(x)\right]^{\frac{1}{k}} + \left[\Gamma_p^{(n)}(y)\right]^{\frac{1}{k}}, \tag{19}$$

holds for $x > 0$, $y > 0$, $k \geq 1$ and $m, n \in \{2s : s \in \mathbb{N}_0\}$.

Proof. We apply the Minkowski's inequality and also use the fact that $A^k + B^k \leq (A+B)^k$, for $A, B \geq 0$ and $k \geq 1$. By (3), we obtain

$$\begin{aligned} &\left[\Gamma_p^{(m)}(x) + \Gamma_p^{(n)}(y)\right]^{\frac{1}{k}} \\ &= \left[\int_0^\infty (\ln t)^m t^{x-1} e^{\left(-t-\frac{p}{t}\right)} dt + \int_0^\infty (\ln t)^n t^{y-1} e^{\left(-t-\frac{p}{t}\right)} dt\right]^{\frac{1}{k}} \\ &= \left[\int_0^\infty \left(\left[(\ln t)^{\frac{m}{k}} t^{\frac{x-1}{k}} e^{\frac{1}{k}\left(-t-\frac{p}{t}\right)}\right]^k + \left[(\ln t)^{\frac{n}{k}} t^{\frac{y-1}{k}} e^{\frac{1}{k}\left(-t-\frac{p}{t}\right)}\right]^k\right) dt\right]^{\frac{1}{k}} \\ &\leq \left[\int_0^\infty \left(\left[(\ln t)^{\frac{m}{k}} t^{\frac{x-1}{k}} e^{\frac{1}{k}\left(-t-\frac{p}{t}\right)}\right] + \left[(\ln t)^{\frac{n}{k}} t^{\frac{y-1}{k}} e^{\frac{1}{k}\left(-t-\frac{p}{t}\right)}\right]\right)^k dt\right]^{\frac{1}{k}} \\ &\leq \left[\int_0^\infty (\ln t)^m t^{x-1} e^{\left(-t-\frac{p}{t}\right)} dt\right]^{\frac{1}{k}} + \left[\int_0^\infty (\ln t)^n t^{y-1} e^{\left(-t-\frac{p}{t}\right)} dt\right]^{\frac{1}{k}} \\ &= \left[\Gamma_p^{(m)}(x)\right]^{\frac{1}{k}} + \left[\Gamma_p^{(n)}(y)\right]^{\frac{1}{k}}, \end{aligned}$$

which gives the desired result. □

Theorem 3.13. *The inequality*

$$\exp\left\{\Gamma_p^{(m-r)}(x)\right\}\exp\left\{\Gamma_p^{(m+r)}(x)\right\} \geq \left(\exp\left\{\Gamma_p^{(m)}(x)\right\}\right)^2, \tag{20}$$

holds for $x > 0$ and $m, r \in \{2s : s \in \mathbb{N}_0\}$ such that $m \geq r$.

Proof. By using (3), we make the following estimation.

$$\begin{aligned}
&\frac{1}{2}\left(\Gamma_p^{(m-r)}(x) + \Gamma_p^{(m+r)}(x)\right) - \Gamma_p^{(m)}(x) \\
&= \frac{1}{2}\left(\int_0^\infty (\ln t)^{m-r} t^{x-1} e^{\left(-t-\frac{p}{t}\right)} dt + \int_0^\infty (\ln t)^{m+r} t^{x-1} e^{\left(-t-\frac{p}{t}\right)} dt\right) \\
&\quad - \int_0^\infty (\ln t)^m t^{x-1} e^{\left(-t-\frac{p}{t}\right)} dt \\
&= \frac{1}{2}\int_0^\infty \left[\frac{1}{(\ln t)^r} + (\ln t)^r - 2\right] (\ln t)^m t^{x-1} e^{\left(-t-\frac{p}{t}\right)} dt \\
&= \frac{1}{2}\int_0^\infty \left[1 - (\ln t)^r\right]^2 (\ln t)^{m-r} t^{x-1} e^{\left(-t-\frac{p}{t}\right)} dt \\
&\geq 0.
\end{aligned}$$

Thus,

$$\Gamma_p^{(m-r)}(x) + \Gamma_p^{(m+r)}(x) \geq 2\Gamma_p^{(m)}(x),$$

and by taking exponents, we obtain (20). □

Remark 3.14. Theorem 3.13 is a generalization of the results of Mortici as given in Theorem 3.1 of [29].

Remark 3.15. For similar results concerning other generalizations and other special functions, one may refer to the recent works [30], [31], [32], [33], [34], [35] and [36].

4 Conclusion

In this work, we have established some inequalities involving the Chaudhry-Zubair extension of the gamma function. The established results are generalizations of some known results in the literature. We anticipate that, the results of this paper will trigger a new research direction for further studies of the function.

Competing Interests

The authors declare that there is no competing interests regarding the publication of this paper.

Acknowledgement

The authors are grateful to the anonymous referees for thorough reading of the manuscript.

References

[1] Abramowitz M, Stegun IA, (Eds.). Handbook of mathematical functions with formulas, graphs, and mathematical tables. National Bureau of Standards, Washington; 1970.

[2] Chaudhry MA, Zubair SM, Generalized incomplete gamma functions with applications. Journal of Computational and Applied Mathematics. 1994;55:99-124.

[3] Díaz R, Pariguan E, On hypergeometric functions and Pochhammer k-symbol. Divulg. Mat. 2007;15:179-192.

[4] Díaz R, Teruel C. q, k-Generalized gamma and beta functions. Journal of Nonlinear Mathematical Physics. 2005;12(1):118-134.

[5] Gasper G, Rahman M. Basic hypergeometric series. Cambridge University Press; 1990.

[6] Krasniqi VB, Srivastava HM, Dragomir SS. Some complete monotonicity properties for the (p, q)-gamma function. Applied Mathematics and Computation. 2013;219(21):10538-10547.

[7] Lahcene B, Mohammed A. On a five and six parameters generalization of the Gamma function. Theoretical Mathematics & Applications. 2017;7(1) 1-14.

[8] Loc TG, Tai TD. The generalized gamma functions. Acta Mathematica Vietnamica. 2012;37(2):219-230.

[9] Mathai AM, Haubold HJ. Special functions for Applied Scientists. Springer; 2008.

[10] Nantomah K, Prempeh E, Twum SB. On a (p, k)-analogue of the Gamma function and some associated Inequalities. Moroccan Journal of Pure and Applied Analysis. 2016;2(2):79-90.

[11] Rahman G, Nisar KS, Choi J, Mubeen S, Arshad M. Extensions of the real matrix-variate gamma and beta functions and their applications. Far East Journal of Mathematical Sciences. 2017;101(10):2333-2347.

[12] Chaudhry MA, Qadir A, Rafique M, Zubair SM. Extension of euler's beta function. Journal of Computational and Applied Mathematics. 1997;78(1):19-32.

[13] Chaudhry MA, Zubair SM. Extended incomplete functions with application. Journal of Mathematical Analysis and Applications. 2002;274(2):725-745.

[14] Choi J, Rathie AK, Parmar RK. Extension of extended beta, hypergeometric and confluent hypergeometric functions. Honam Mathematical Journal. 2014;36(2):357-385.

[15] Khan S, Ai-Saad M. Euler typer integrals and integrals in terms of extended beta function. Journal of Mathematics. 2014;12. Article ID 686324.

[16] Khan WA, Nisar KS, Ahmad M. Euler type integral operator involving k-mittag-leffler function; 2017. arXiv:1704.09001v1 [math.CA].

[17] Luo M, Raina R. Extended generalized hypergeometric functions and their applications. Bulletin of Mathematical Analysis and Applications. 2013;5(4):65-77.

[18] Miller A, Moskowitz I. On certain incomplete gamma function. Journal of Computational and Applied Mathematics. 1998;91(2):179-190.

[19] Mubeen S, Nisar KS, Rahman G. Inequalities of extended (p,q)-beta and confluent hupergeometric function; 2017. arXiv:1703.08852v2 [math.CA] .

[20] Mubeen S, Purohit SD, Arshad M, Rahman G. Extension of k-gamma, k-beta function and k-beta distribution. Journal of Mathematecal Analysis. 2016;7(5):118-131.

[21] Nagar DK, Gomez-Noguera SA, Gupta AK. Generalized extended matrix variate beta and gamma functions and their applications. Ingeniera y Ciencia. 2016;12(24):51-82.

[22] Ozergin E, Ozarslan MA, Altin A. Extension of gamma, beta and hypergeometric function. Journal of Computational and Applied Mathematics. 2011;235(16):4601-4610.

[23] Parmar RK. A new geralization of gamma, beta, hypergeometric and confluent hypergeometric functions. Le Matematiche. 2013;LXVIII-Fasc.II:33-52.

[24] Kazarinoff ND. Analytic Inequalities. Holt, Rinehart and Winston. New York; 1961.

[25] Mitrinovic DS. Analytic Inequalities. Springer-Verlag. New York; 1970.

[26] Nantomah K. Generalized holders and minkowskis inequalities for jacksons q-integral and some applications to the incomplete q-gamma function. Abstract and Applied Analysis. 2017;6. Article ID 9796873.

[27] Mortici C. New inequalities for some special functions via the cauchy-buniakovsky-schwarz inequality. Tamkang Journal of Mathematics. 2011;42(1):53-57.

[28] Nisar KS, Qi F, Rahman G, Mubeen S, Arshad M. Some inequalities involving the extended gamma function and the Kummer confluent hypergeometric k-function. Journal of Inequalities and Applications. 2018;12. Art ID:135.

[29] Mortici C. Turan-type inequalities for the gamma and polygamma functions. Acta Universitatis Apulensis. 2010;23:117-121.

[30] Nantomah K. On some properties and inequalities for the Nielsen's β-function. SCIENTIA Series A: Mathematical Sciences. 2017-2018;28:43-54.

[31] Nantomah K. Remarks on some inequalities for analogues of the polygamma function. Mathematical Sciences and Applications E-Notes. 2018;6(1):93-98.

[32] Nantomah K, Iddrisu MM, Okpoti CA. On a q-analogue of the Nielsen's β-Function. International Journal of Mathematics And its Applications. 2018;6(2-A):163-171.

[33] Nantomah K, Merovci F, Nasiru S. A generalization of the exponential integral and some associated inequalities. Honam Mathematical Journal. 2017;39(1):49-59.

[34] Nantomah K, Nasiru S. Inequalities for the m-th derivative of the (q, k)-Gamma function. Moroccan Journal of Pure and Applied Analysis. 2017;3(1):63-69.

[35] Nantomah K, Nisar KS, Gehlot KS. On a k-extension of the Nielsen's β-function. International Journal of Nonlinear Analysis and Applications. 2018;9(2):191-201.

[36] Nantomah K, Prempeh E, Twum SB. Inequalities involving derivatives of the (p, k)-gamma function. Konuralp Journal of Mathematics. 2017;5(1):232-239.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://www.sdiarticle3.com/review-history/49320