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Stability Analysis of Multi-Infections (Malaria, Zika-Virus and Elephantiasis) Model

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Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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Abstract

The paper presents a multi-infections system model to study the transmission dynamics of Malaria, Zika-Virus and Elephantiasis in an endemic region such as Kedougou in the Southeastern part of Senegal and other parts of the world where it is possible to have multi-infections of the three diseases simultaneously. We performed the disease-free equilibrium and it is shown to be globally asymptotically stable when the associated threshold known as the basic reproduction number for the model is less than unity. Investigation on the existence and stability of equilibria is also performed, the model is found to exhibit backward bifurcation so that for R_0 less than unity is not sufficient to eradicate the disease from the population and there is the need to lower R_0 below a certain threshold for effective disease control. Sensitivity analysis is performed to determine parameters that have high influence on the basic reproduction number.

Keywords: Multi-infections; stability analysis; bifurcation analysis etc.

1 Introduction

Zika virus disease is caused by a virus which is transmitted mostly by female *Aedes aegypti* mosquito [1] which is also responsible for the transmission of chikungunya and dengue fever. The incidence of Zika virus disease is spreading and this is partly due to the fact that there is neither cure nor vaccine. Malaria which is

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also a vector borne disease is caused by a female Anopheles mosquito. There are four species of parasites namely: *Plasmodium falciparum, Plasmodium vivax, Plasmodium ovale* and *Plasmodium malariae* that infect human. The disease continues to be a major problem in sub-Sahara Africa, Asia, Central and South America and the Middle East. Almost 40% of the world's population settles in endemic areas [2]. Elephantiasis which is also known as Lymphatic filariasis and also a vector borne disease is a neglected tropical disease which is caused by parasitic worms that are spread by the bite of mosquitoes [3]. There are three types of worms namely; Wuchereria Bancroft, Brugia Malayi and Brugia Timori with Wuchereria accounting for 90% of the cases and is found almost in all the tropical and subtropical areas around the globe [4]. The major differences in the manner we combat malaria and zika virus diseases generate from the fact that they are spread by two different types of mosquitoes namely female anopheles for malaria and female *Aedes aegypti* for zika virus. Female anopheles mosquito bite at night and at dawn and can transmit the malaria; whiles, the *Aedes aegypti* mosquito bite during the day time [5] which makes prevention more complex to deal with. Elephantiasis can however be transmitted through the bite of infected female Anopheles mosquitoes, Aedes mosquitoes and Culux depending on the geographical location of the vector. Although, elephantiasis andzika virus do not kill like malaria does but the stigmatization attached to the disability or deformity in the case of elephantiasis and the microcephaly which is linked to zika-virusmakethem more embarrassing.

In this research, the idea is to propose a mathematical model for the transmission dynamics of multi-infections (Malaria-Elephantiasis-Zika) without control. A considerable amount of research, has been done on co-infections of various diseases, however, there is none on this multi-infections (Malaria-Elephantiasis-Zika).

Section 2 discusses the model formulation, Section 3 handles the sufficient conditions for local and global stability of the disease-free equilibrium and endemic equilibria. Section 4 discuss the backward bifurcation phenomenon and in section 5 we present the:

1.1 Model formulation and description

The new proposed model subdivides human population into eleven compartments namely: susceptible individual (S_h), infectious Malaria (I_m), infectious Elephantiasis(I_f), infectious Zika (I_z), both infectious Malaria and Zika (I_{mz}), both infectious Malaria and Elephantiasis (I_{mf}), both infectious Zika and Elephantiasis (I_{zf}), infectious Malaria, Zika and Elephantiasis(I_{mzf}) called multi-infections, recovered from Malaria only, (R_m), recovered from Zika only (R_z), a recovered from Elephantiasis only(R_f). The female mosquito population is also partitioned into four compartments, that is, susceptible mosquitoes (S_m), female Anopheles mosquitoes infected with parasite(I_p), female *Aedes aegypti* mosquito infected with Zika virus (I_a) and female Culux mosquito infected with a roundworm(I_w).

Susceptible humans (S_h) get infected with malaria infection through a bite from infectious female Anopheles mosquitoes with malaria parasite at the rateδ_m. Zika infection is also acquired through a bite of infectious female *Aedes aegypti* mosquito with Zika virus at the rateδ_z, and finally, Elephantiasis infection occurs through a bite of infectious female mosquito with worm infection at the rateδ_f. Human beings are recruited into their population at the rate Π_h and the mosquitoes are also recruited at the rate Π_m Individuals can exit both populations through natural death rates μ_h and μ_m respectively for humans and mosquitoes. Humans can also exit through malaria induced death rate η. For co-infections, individual with malaria infection can either get Zika infection when bitten by infectious female aedes mosquitoes at the rate $\delta_z$or elephantiasis infection through a bite of infectious female Culux mosquitoes with worm infection at the rate δ_f. Also, someone with Zika infection can either get malaria infection when bitten by infectious anopheles mosquitoes at the rate δ_m or elephantiasis infection through a bite of infectious mosquitoes with worm infection at the rate δ_f. And finally, a person with elephantiasis infection can either get malaria infection when bitten by infectious anopheles mosquitoes at the rate $\delta_m$or Zika infection through a bite of infectious aedes mosquitoes with Zika virus at the rate δ_z. Our attention is now turned to the multi-infections an individual with both malaria and Zika infections can get; the third infection which is elephantiasis at the rateδ_f. It is also possible to have a situation where a co-infected person with malaria and elephantiasis

infections can get the Zika infection at the rate δ_z to complete his or her multi-infections. To complete the multi-infections, a co-infected individual with Zika and elephantiasis infections can get the malaria infection at the rate δ_m.

For the mosquito population, we have infectious anopheles mosquito when the susceptible anopheles mosquito bites someone with malaria parasite at the rate δ_m. Infectious aedes mosquitoes also come into existence when a susceptible aedes mosquito bites a person with Zika virus at the rate δ_z. In addition, an infectious mosquito with worm infection occurs when a susceptible mosquito bites someone with elephantiasis worm at the rate δ_f.

Tables one and two below describe the state variables and the parameters respectively.

Table 1. State variables and description

State variables	Description of the state variable
$N_h(t)$	Total human population.
$N_m(t)$	Total mosquito population.
$S_h(t)$	Susceptible human.
$I_m(t)$	Individual infected with malaria.
$I_z(t)$	Individual infected with Zika.
$I_f(t)$	Individual infected with Elephantiasis.
$I_{mz}(t)$	Individual infected with Malaria and Elephantiasis.
$I_{mf}(t)$	Individual infected with Malaria and Elephantiasis
$I_{zf}(t)$	Individual infected with Zika and Elephantiasis.
$I_{mzf}(t)$-	Individual infected with multi-infections; Malaria-Zika- Elephantiasis.
$R_m(t)$	Individual recovered from malaria.
$R_m(t)$	Individual recovered from Zika.
$R_m(t)$	Individual recovered from Elephantiasis.
$S_m(t)$	Susceptible mosquito.
$I_p(t)$	Mosquito infected with parasite.
$I_a(t)$	Mosquito infected with the zika virus.
$I_w(t)$	Mosquito infected with worms.

Table 2. Description of parameters used in the model

Parameters	Description of parameters
α_m	Rate at which one acquires malaria infection.
α_f	Rate at which one acquires elephantiasis infection.
α_z	Rate at which one acquires Zika virus infection.
τ_m	Rate at which one recovers from malaria only.
τ_z	Rate at which one recovers from Zika only.
τ_f	Rate at which one recovers from elephantiasis only.
ψ	Rate at which individual returns to susceptible after recovery from malaria.
φ	Rate at which individual returns to susceptible after recovery from elephantiasis.
ϑ_m	Probability of transmission of infection from a mosquito with parasite infection to a susceptible human given that there is contact between the two.
δ_m	The number of bites of humans per Anopheles mosquito per unit time.
ϑ_f	Probability of transmission of infection from a mosquito with worm infection to A susceptible human given that a contact rate between the two occurs.
δ_f	The number of bites of humans per Culex mosquito per unit time.
ϑ_z	Probability of transmission of infection from a mosquito with zika virus to susceptible human given that a contact between the two occur
δ_z	The number of bites of humans per *Aedes aegypti* mosquito per unit time.

2 Multi-infections Model (Malaria Elephantiasis Zika)

$$\left.\begin{aligned}
\frac{dS_h}{dt} &= \Pi_h + \psi R_m + \varphi R_f - \alpha_m S_h - \mu_h S_h - \alpha_z S_h - \alpha_f S_h \\
\frac{dI_m}{dt} &= \alpha_m S_h - \tau_m I_m - (\mu_h + \eta) I_m + \tau_f I_{mf} - \alpha_f I_m + \tau_z I_{mz} - \alpha_z I_m \\
\frac{dI_f}{dt} &= \alpha_f S_h - \mu_h I_f - \alpha_m I_f + \tau_m I_{mf} - \tau_f I_f + \tau_z I_{zf} - \alpha_z I_f \\
\frac{dI_z}{dt} &= \alpha_z S_h - \tau_z I_z - \alpha_m I_z + \tau_m I_{mz} + \tau_f I_{zf} - \alpha_f I_z - \mu_h I_z \\
\frac{dI_{mz}}{dt} &= \alpha_z I_m - \tau_z I_{mz} - (\mu_h + \eta) I_{mz} + \tau_f I_{mzf} - \alpha_f I_{mz} + \alpha_m I_z - \tau_m I_{mz} \\
\frac{dI_{mf}}{dt} &= \alpha_f I_m - \tau_f I_{mf} - (\mu_h + \eta) I_{mf} + \tau_z I_{mzf} - \alpha_z I_{mf} + \alpha_m I_f - \tau_m I_{mf} \\
\frac{dI_{zf}}{dt} &= \alpha_z I_f - \tau_f I_{zf} - \mu_h I_{zf} + \tau_m I_{mzf} - \alpha_m I_{zf} + \alpha_f I_z - \tau_z I_{zf} \\
\frac{dI_{mzf}}{dt} &= \alpha_m I_{zf} - \tau_m I_{mzf} - (\mu_h + \eta) I_{mzf} - \tau_f I_{mzf} + \alpha_f I_{mz} + \alpha_z I_{mf} - \tau_z I_{mzf} \\
\frac{dR_m}{dt} &= \tau_m I_m - (\mu_h + \psi) R_m \\
\frac{dR_f}{dt} &= \tau_f I_f - (\mu_h + \varphi) R_f \\
\frac{dR_z}{dt} &= \tau_z I_z - \mu_h R_z \\
\frac{dS_m}{dt} &= \Pi_m - \mu_m S_m - \lambda_p S_m - \lambda_w S_m - \lambda_a S_m \\
\frac{dI_p}{dt} &= \lambda_p S_m - \mu_m I_p \\
\frac{dI_w}{dt} &= \lambda_w S_m - \mu_m I_w \\
\frac{dI_a}{dt} &= \lambda_a S_m - \mu_m I_a
\end{aligned}\right\}. \qquad (1)$$

$$\left.\begin{aligned}
\alpha_m &= \frac{\vartheta_m \delta_m I_p}{N_h}, \alpha_f = \frac{\vartheta_f \delta_f I_w}{N_h}, \alpha_z = \frac{\vartheta_z \delta_z I_a}{N_h} \\
\lambda_p &= \frac{\vartheta_m \delta_m (I_m + I_{mz} + I_{mf} + I_{mzf})}{N_h} S_m, \lambda_w = \frac{\vartheta_f \delta_f (I_f + I_{mf} + I_{zf} + I_{mzf})}{N_h} S_m, \\
\lambda_a &= \frac{\vartheta_z \delta_z (I_z + I_{mz} + I_{zf} + I_{mzf})}{N_h} S_m
\end{aligned}\right\} \qquad (2)$$

In this regard, the researcher tries to come out with the basic results concerning the model (1).The following theorem shows the region within which the model will be examined in the subsequent sections.

Theorem 1: If the initial state variables are non-negative i.e.

$$\left\{\begin{matrix}\left(S_h(0), I_m(0), I_f(0), I_z(0), I_{mz}(0), I_{mf}(0), I_{fz}(0), I_{mfz}(0), R_m(0), R_f(0), R_z(0), S_m(0), I_p(0), I_w(0), I_a(0)\right)\\ \geq 0\end{matrix}\right\}$$

Then the solution set

$$\left\{\begin{matrix}S_h(t) \geq 0, S_m(t) \geq 0, I_m(t) \geq 0, I_f(t) \geq 0, I_z(t) \geq 0, I_{mz}(t) \geq 0, I_{mf}(t) \geq 0,\\ I_{fz}(t) \geq 0, I_{mfz}(t) \geq 0, I_p(t) \geq 0, I_w(t) \geq 0, I_a(t) \geq 0, R_m(t) \geq 0, R_f(t) \geq 0, R_z(t) \geq 0\end{matrix}\right\}(t)$$ of the system (1) is positive forall$t \geq 0$. Moreover$lim_{t \leftarrow \infty}\, sup\, N_h(t) \leq \frac{\Pi_h}{\mu_h}$ and $lim\ sup\, N_m\,(t) \leq \frac{\Pi_m}{\mu_m}$.

Again, if $N_h(0) \leq \frac{\Pi_h}{\mu_h}$ and $N_m(0) \leq \frac{\Pi_m}{\mu_m}$ then $N_h(t) \leq \frac{\Pi_h}{\mu_h}$and $N_m(t) \leq \frac{\Pi_m}{\mu_m}$.

More importantly, the region

$$\Omega_h = \left\{\left(S_h, I_m, I_f, I_z, I_{mz}, I_{mf}, I_{fz}, I_{mfz}, R_m, R_f, R_z\right) \in R_+^{11} : N_h(t) \leq \frac{\Pi_h}{\mu_h}\right\}$$

$$\Omega_m = \left(S_m, I_p, I_w, I_a\right) \in R_+^4, N_m(t) \leq \frac{\Pi_m}{\mu_m}$$

Is positively invariant. The theorem 1 above indicates that the model (1) is biologically and epidemiologically well posed in the region and thus, the dynamics of the model can be sufficiently studied inΩ [6,7].

The multi-infections model can be divided into various sub-models namely Malaria – Zika co-infections model, Malaria –Elephantiasis co-infections model, Zika- Elephantiasis co-infection model, Malaria only model, Zika-virus only model and Elephantiasis only model. The sub-models are given as follows

2.1 Co-infection Malaria- Zika Sub-model

$$\left.\begin{aligned}
\frac{dS_h}{dt} &= \Pi_h - \mu_h S_h - \alpha_m S_h - \alpha_z S_h + \psi R_m\\
\frac{dI_m}{dt} &= \alpha_m S_h - \tau_m I_m - (\mu_h + \eta) I_m - \alpha_z I_m + \tau_z I_{mz}\\
\frac{dI_z}{dt} &= \alpha_z S_h - \tau_z I_z - \alpha_m I_z + \tau_m I_{mz} - \mu_h I_z\\
\frac{dI_{mz}}{dt} &= \alpha_z I_m - \tau_z I_{mz} - (\mu_h + \eta) I_{mz} + \alpha_m I_z - \tau_m I_{mz}\\
\frac{dR_m}{dt} &= \tau_m I_m - (\mu_h + \psi) R_m\\
\frac{dR_z}{dt} &= \tau_z I_z - \mu_h R_z\\
\frac{dS_m}{dt} &= \Pi_m - \mu_m S_m - \lambda_p S_m - \lambda_a S_m\\
\frac{dI_p}{dt} &= \lambda_p S_m - \mu_m I_p\\
\frac{dI_a}{dt} &= \lambda_a S_m - \mu_m I_a
\end{aligned}\right\} \quad (3)$$

$$\alpha_m = \frac{\vartheta_m \delta_m I_p}{N_h}, \alpha_z = \frac{\vartheta_z \delta_z I_a}{N_h}, \lambda_p = \frac{\vartheta_m \delta_m (I_m + I_{mz})}{N_h}, \lambda_a = \frac{\vartheta_z \delta_z (I_z + I_{mz})}{N_h} \quad (4)$$

2.2 Co-infection model malaria- elephantiasis

$$\left.\begin{aligned}
\frac{dS_h}{dt} &= \Pi_h + \psi R_m + \varphi R_f - \alpha_m S_h - \mu_h S_h - \alpha_f S_h \\
\frac{dI_m}{dt} &= \alpha_m S_h - \tau_m I_m - (\mu_h + \eta) I_m - \alpha_f I_m + \tau_f I_{mf} \\
\frac{dI_f}{dt} &= \alpha_f S_h - \mu_h I_f - \alpha_m I_f - \tau_f I_f + \tau_m I_{mf} \\
\frac{dI_{mf}}{dt} &= \alpha_f I_m - \left(\mu_h + \eta + \tau_m + \tau_f\right) I_{mf} + \alpha_m I_f \\
\frac{dR_m}{dt} &= \tau_m I_m - (\mu_h + \psi) R_m \\
\frac{dR_f}{dt} &= \tau_f I_f - (\mu_h + \varphi) R_f \\
\frac{dS_m}{dt} &= \Pi_m - \mu_m S_m - \lambda_p S_m - \lambda_w S_m \\
\frac{dI_p}{dt} &= \lambda_p S_m - \mu_m I_p \\
\frac{dI_w}{dt} &= \lambda_w S_m - \mu_m I_w
\end{aligned}\right\} \quad (5)$$

$$\alpha_m = \frac{\vartheta_m \delta_m I_p}{N_h}, \alpha_f = \frac{\vartheta_f \delta_f I_w}{N_h}, \lambda_p = \frac{\vartheta_m \delta_m \left(I_m + I_{mf}\right)}{N_h}, \lambda_w = \frac{\vartheta_f \delta_f \left(I_f + I_{mf}\right)}{N_h} \quad (6)$$

2.3 Co-infection Zika- Elephantiasis

$$\left.\begin{aligned}
\tfrac{dS_h}{dt} &= \Pi_h - \mu_h S_h - \alpha_z S_h - \alpha_f S_h + \varphi R_f \\
\tfrac{dI_z}{dt} &= \alpha_z S_h - \tau_z I_z - \alpha_f I_z + \tau_f I_{zf} - \mu_h I_z \\
\tfrac{dI_f}{dt} &= \alpha_f S_h - \mu_h I_f - \tau_f I_f + \tau_z I_{zf} - \alpha_z I_f \\
\tfrac{dI_{zf}}{dt} &= \alpha_z I_f - \tau_f I_{zf} - \mu_h I_{zf} + \alpha_f I_z - \tau_z I_{zf} \\
\tfrac{dR_z}{dt} &= \tau_z I_z - \mu_h R_z \\
\tfrac{dR_f}{dt} &= \tau_f I_f - (\mu_h + \varphi) R_f \\
\tfrac{dS_m}{dt} &= \Pi_m - \mu_m S_m - \lambda_w S_m - \lambda_a S_m \\
\tfrac{dI_a}{dt} &= \lambda_a S_m - \mu_m I_a \\
\tfrac{dI_w}{dt} &= \lambda_w S_m - \mu_m I_w
\end{aligned}\right\} \quad (7)$$

$$\alpha_z = \frac{\vartheta_z \delta_z I_a}{N_h}, \alpha_f = \frac{\vartheta_f \delta_f I_w}{N_h}, \lambda_a = \frac{\vartheta_z \delta_z \left(I_z + I_{zf}\right)}{N_h}, \lambda_w = \frac{\vartheta_f \delta_f \left(I_f + I_{zf}\right)}{N_h} \quad (8)$$

2.4 Malaria only model

$$\left.\begin{aligned}
\frac{dS_h}{dt} &= \Pi_h + \psi R_m - \alpha_m S_h - \mu_h S_h \\
\frac{\boldsymbol{dI_m}}{\boldsymbol{dt}} &= \alpha_m S_h - \tau_m I_m - (\mu_h + \eta) I_m \\
\frac{dR_m}{dt} &= \tau_m I_m - (\mu_h + \psi) R_m \\
\frac{dS_m}{dt} &= \Pi_m - \mu_m S_m - \lambda_p S_m \\
\frac{dI_p}{dt} &= \lambda_p S_m - \mu_m I_p
\end{aligned}\right\} \quad (9)$$

$$\alpha_m = \frac{\vartheta_m \delta_m I_p}{N_h}, \lambda_p = \frac{\vartheta_m \delta_m I_m}{N_h} S_m \quad (10)$$

2.5 Zika-Virus only model

$$\left.\begin{aligned}
\frac{dS_h}{dt} &= \Pi_h - \mu_h S_h - \alpha_z S_h \\
\frac{dI_z}{dt} &= \alpha_z S_h - \tau_z I_z - \mu_h I_z \\
\frac{dR_z}{dt} &= \tau_z I_z - \mu_h R_z \\
\frac{dS_m}{dt} &= \Pi_m - \mu_m S_m - \lambda_a S_m \\
\frac{dI_a}{dt} &= \lambda_a S_m - \mu_m I_a
\end{aligned}\right\} \quad (11)$$

$$\alpha_z = \frac{\vartheta_z \delta_z I_a}{N_h}, \lambda_a = \frac{\vartheta_z \delta_z I_z}{N_h} S_m \quad (12)$$

2.6 Elephantiasis model only

$$\left.\begin{aligned}
\frac{dS_h}{dt} &= \Pi_h - \mu_h S_h - \alpha_f S_h \\
\frac{dI_f}{dt} &= \alpha_f S_h - \mu_h I_f - \tau_f I_f \\
\frac{dR_f}{dt} &= \tau_f I_f - \mu_h R_f \\
\frac{dS_m}{dt} &= \Pi_m - \mu_m S_m - \lambda_w S_m \\
\frac{dI_w}{dt} &= \lambda_w S_m - \mu_m I_w
\end{aligned}\right\} \quad (13)$$

$$\alpha_f = \frac{\vartheta_f \delta_f I_w}{N_h}, \lambda_w = \frac{\vartheta_f \delta_f I_f}{N_h} S_m \quad (14)$$

3 Stability of the Disease Free-Equilibrium (DFE)

This is the steady state solution where there is no infection in the population. The disease-free equilibrium of the model are stated in the subsection below.

3.1 Stability of the Disease Free-equilibrium (DFE) of Multi-infections model

The (DFE) of the multi-infections (Malaria, Zika virus and Elephantiasis) model is obtained when the right hand side of equation (1) is set to zero. That is

$$\{I_m = I_f = I_z = I_{mz} = I_{mf} = I_{fz} = I_{mfz} = I_p = I_w = I_a = R_m = R_f = R_z = 0\}$$

Hence the disease free equilibrium point at the multi-infections is given as

$$\left.\begin{aligned}\mathcal{E}_{mfz} &= \left(S_h, I_m, I_f, I_z, I_{mz}, I_{mf}, I_{fz}, I_{mfz}, R_m, R_f, R_z, S_m, I_p, I_w, I_a\right)\\ &= \left(\frac{\Pi_h}{\mu_h}0,0,0,0,0,0,0,0,0,0,\frac{\Pi_m}{\mu_m}0,0,0\right)\end{aligned}\right\} \quad (15)$$

3.1.1 Basic reproduction number f

Using the next generation matrix, the basic reproduction number can be defined by the average number of secondary infectious cases produced by a single infective individual in a population where everyone is susceptible. Applying the next generation matrix method of [8] and the basic reproduction number of multi-infections is the spectral radius of the matrix $\mathcal{F}V^{-1}$. Where $\mathcal{F}$ and V are the transmission and transition matrices respectively given by

$$F=\begin{vmatrix} -a_{11} & 0 & 0 & 0 & 0 & 0 & 0 & a_{18} & -a_{19} & -a_{20} \\ 0 & -a_{22} & 0 & 0 & 0 & 0 & 0 & -a_{28} & a_{29} & -a_{30} \\ 0 & 0 & -a_{33} & 0 & 0 & 0 & 0 & -a_{38} & -a_{39} & a_{40} \\ a_{41} & 0 & a_{43} & -a_{44} & 0 & 0 & 0 & a_{38} & -a_{49} & a_{20} \\ a_{44} & a_{43} & 0 & 0 & -a_{41} & 0 & 0 & a_{28} & a_{19} & -a_{60} \\ 0 & a_{41} & a_{44} & 0 & 0 & -a_{43} & 0 & -a_{68} & a_{39} & a_{30} \\ 0 & 0 & 0 & a_{44} & a_{41} & a_{43} & 0 & a_{68} & a_{49} & a_{60} \\ a_{81} & 0 & 0 & a_{84} & a_{85} & 0 & a_{87} & 0 & 0 & 0 \\ 0 & a_{92} & 0 & 0 & a_{95} & a_{96} & a_{97} & 0 & 0 & 0 \\ 0 & 0 & a_{103} & a_{104} & 0 & a_{106} & a_{107} & 0 & 0 & 0 \end{vmatrix}$$

$$a_{11} = -\left(\frac{\vartheta_f\delta_f I_w}{N_h} + \frac{\vartheta_z\delta_z I_a}{N_h}\right), a_{18} = \frac{\vartheta_m\delta_m S_h}{N_h}, a_{19} = -\frac{\vartheta_f\delta_f I_m}{N_h}, a_{20} = -\frac{\vartheta_z\delta_z I_m}{N_h}$$

$$a_{22} = -\left(\frac{\vartheta_m\delta_m I_p}{N_h} + \frac{\vartheta_z\delta_z I_a}{N_h}\right), a_{28} = -\frac{\vartheta_m\delta_m I_f}{N_h}, a_{29} = \frac{\vartheta_f\delta_f S_h}{N_h}, a_{30} = -\frac{\vartheta_z\delta_z I_f}{N_h},$$

$$a_{33} = -\left(\frac{\vartheta_m\delta_m I_p}{N_h} + \frac{\vartheta_f\delta_f I_w}{N_h}\right), a_{38} = -\frac{\vartheta_m\delta_m I_z}{N_h}, a_{39} = -\frac{\vartheta_f\delta_f I_z}{N_h}, a_{40} = \frac{\vartheta_z\delta_z S_h}{N_h}, a_{43} = \frac{\vartheta_m\delta_m I_p}{N_h},$$

$$a_{41} = \frac{\vartheta_z\delta_z I_a}{N_h}, a_{43} = \frac{\vartheta_m\delta_m I_p}{N_h}, a_{44} = -\frac{\vartheta_f\delta_f I_w}{N_h}, a_{49} = -\frac{\vartheta_f\delta_f I_{mz}}{N_h}, a_{60} = -\frac{\vartheta_z\delta_z I_{mf}}{N_h},$$

$$a_{78} = \frac{\vartheta_m\delta_m I_{zf}}{N_h} a_{81} = \frac{\vartheta_m\delta_m\left(I_{mz} + I_{mf} + I_{mzf}\right)}{N_h} S_m, a_{84} = \frac{\vartheta_m\delta_m\left(I_m + I_{mf} + I_{mzf}\right)}{N_h} S_m,$$

$$a_{85} = \frac{\vartheta_m\delta_m(I_m+I_{mz}+I_{mzf})}{N_h} S_m, a_{87} = \frac{\vartheta_m\delta_m(I_m+I_{mz}+I_{mf})}{N_h} S_m, a_{92} = \frac{\vartheta_f\delta_f(I_{mf}+I_{zf}+I_{mzf})}{N_h} S_m,$$

$$a_{95} = \frac{\vartheta_f\delta_f(I_f+I_{zf}+I_{mzf})}{N_h} S_m, a_{96} = \frac{\vartheta_f\delta_f(I_f+I_{mf}+I_{mzf})}{N_h} S_m, a_{97} = \frac{\vartheta_f\delta_f(I_f+I_{mf}+I_{zf})}{N_h} S_m a_{103} = \frac{\vartheta_z\delta_z(I_{mz}+I_{zf}+I_{mzf})}{N_h} S_m, a_{104} = \frac{\vartheta_z\delta_z(I_z+I_{zf}+I_{mzf})}{N_h} S_m,$$

$$a_{106} = \frac{\vartheta_z\delta_z\left(I_z + I_{mz} + I_{mzf}\right)}{N_h} S_m, a_{107} = \frac{\vartheta_z\delta_z\left(I_z + I_{mz} + I_{zf}\right)}{N_h} S_m$$

$$V^{-1} = \begin{vmatrix} c_{11} & 0 & 0 & -c_{14} & c_{15} & 0 & c_{17} & 0 & 0 & 0 \\ 0 & c_{22} & 0 & 0 & c_{25} & c_{26} & c_{27} & 0 & 0 & 0 \\ 0 & 0 & c_{33} & c_{34} & 0 & c_{36} & c_{37} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 & c_{47} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 & c_{57} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} & c_{67} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_{77} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{88} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{88} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{88} \end{vmatrix}$$

$$c_{11} = \frac{1}{b_{11}}, c_{14} = \frac{b_{14}}{b_{11}b_{44}}, c_{15} = \frac{b_{15}}{b_{11}b_{55}}, c_{17} = \frac{b_{14}b_{15}b_{44} + b_{14}b_{15}b_{55}}{b_{11}b_{44}b_{55}b_{77}}, c_{22} = \frac{1}{b_{22}},$$

$$c_{25} = \frac{b_{25}}{b_{22}b_{55}}, c_{26} = \frac{b_{14}}{b_{22}b_{66}}, c_{27} = \frac{b_{14}b_{25}b_{55} + b_{14}b_{25}b_{66}}{b_{22}b_{44}b_{66}b_{77}}, c_{33} = \frac{1}{b_{33}}, c_{34} = \frac{b_{25}}{b_{33}b_{44}},$$

$$c_{36} = \frac{b_{15}}{b_{33}b_{66}}, c_{37} = \frac{b_{15}b_{25}b_{44} + b_{15}b_{25}b_{66}}{b_{33}b_{44}b_{66}b_{77}}, c_{44} = \frac{1}{b_{44}}, c_{47} = \frac{b_{15}}{b_{44}b_{77}}, c_{55} = \frac{1}{b_{55}}, c_{57} = \frac{b_{14}}{b_{55}b_{77}},$$

$$c_{66} = \frac{1}{b_{66}}, c_{67} = \frac{b_{25}}{b_{66}b_{77}}, c_{77} = \frac{1}{b_{77}}, c_{88} = \frac{1}{b_{88}}$$

The basic reproduction number R_0 of the multi-infections is given as the spectral radius of the matrix FV^{-1} so that

$$R_0 = \rho(FV^{-1}) = \left\{ \sqrt{\frac{\vartheta_m^2 \delta_m^2 \Pi_m \mu_h}{(\mu_h + \eta + \tau_m)\Pi_h \mu_m^2}}, \sqrt{\frac{\vartheta_z^2 \delta_z^2 \Pi_m \mu_h}{(\mu_h + \tau_z)\Pi_h \mu_m^2}}, \sqrt{\frac{\vartheta_f^2 \delta_f^2 \Pi_m \mu_h}{(\mu_h + \tau_f)\Pi_h \mu_m^2}} \right\} \quad (16)$$

3.1.2 Local stability at the disease free equilibrium for multi-infections model (Malaria-Zika-Elephantiasis)

Theorem 2: The disease-free equilibrium point is locally asymptotically stable if $R_0 < 1$ $\left(R_{ma} < 1, R_f < 1 \; and R_{zv} < 1\right)$ and unstable if $R_0 > 1$.

Proof

$$J = \begin{pmatrix} -b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \psi & \varphi & 0 & 0 & -b_2 & -b_3 & -b_4 \\ 0 & -b_6 & 0 & 0 & \tau_z & \tau_f & 0 & 0 & 0 & 0 & 0 & 0 & b_2 & 0 & 0 \\ 0 & 0 & -b_9 & 0 & 0 & \tau_m & \tau_z & 0 & 0 & 0 & 0 & 0 & 0 & b_3 & 0 \\ 0 & 0 & 0 & -b_{13} & \tau_m & 0 & \tau_f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_4 \\ 0 & 0 & 0 & 0 & -b_{15} & 0 & 0 & \tau_f & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -b_{18} & 0 & \tau_z & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -b_{20} & \tau_m & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -b_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \tau_m & 0 & 0 & 0 & 0 & 0 & 0 & -b_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tau_f & 0 & 0 & 0 & 0 & 0 & 0 & -b_{24} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tau_z & 0 & 0 & 0 & 0 & 0 & 0 & -\mu_h & 0 & 0 & 0 & 0 \\ 0 & -b_{25} & -b_{26} & -b_{27} & -b_{28} & -b_{29} & -b_{30} & -b_{31} & 0 & 0 & 0 & -b_{32} & 0 & 0 & 0 \\ 0 & b_{25} & 0 & 0 & b_{25} & 0 & b_{25} & b_{25} & 0 & 0 & 0 & b_{33} & -\mu_m & 0 & 0 \\ 0 & 0 & b_{26} & 0 & 0 & b_{26} & b_{26} & b_{26} & 0 & 0 & 0 & b_{34} & 0 & -\mu_m & 0 \\ 0 & 0 & 0 & b_{27} & b_{27} & b_{27} & 0 & b_{27} & 0 & 0 & 0 & b_{35} & 0 & 0 & -\mu_m \end{pmatrix} \quad \ldots(17)$$

$$b_1=\mu_h, b_2=\vartheta_m\delta_m, b_3=\vartheta_f\delta_f, b_4=\vartheta_z\delta_z, b_5=0, b_6=\left(\mu_h+\eta+\tau_m\right), b_7=0, b_8=0, b_9=\left(\mu_h+\tau_f\right)$$
$$, b_{10}=0, b_{11}=0, b_{12}=0, b_{13}=\left(\mu_h+\tau_z\right), b_{14}=0, b_{15}=\left(\mu_h+\eta+\tau_m+\tau_z\right), b_{16}=0, b_{17}=0, b_{18}=\left(\mu_h+\eta+\tau_m+\tau_f\right),$$
$$b_{19}=0, b_{20}=\left(\mu_h+\tau_f+\tau_z\right), b_{21}=0, b_{22}=\left(\mu_h+\eta+\tau_m+\tau_f+\tau_z\right), b_{23}=\left(\mu_h+\psi\right), b_{24}=\left(\mu_h+\varphi\right),$$
$$b_{25}=\frac{\vartheta_m\delta_m\Pi_m\mu_h}{\Pi_h\mu_m}, b_{26}=\frac{\vartheta_f\delta_f\Pi_m\mu_h}{\Pi_h\mu_m}, b_{27}=\frac{\vartheta_z\delta_z\Pi_m\mu_h}{\Pi_h\mu_m}, b_{28}=\frac{\Pi_m\mu_h}{\Pi_h\mu_m}\left(\vartheta_m\delta_m+\vartheta_z\delta_z\right), b_{29}=\frac{\Pi_m\mu_h}{\Pi_h\mu_m}\left(\vartheta_m\delta_m+\vartheta_f\delta_f\right),$$
$$b_{30}=\frac{\Pi_m\mu_h}{\Pi_h\mu_m}\left(\vartheta_z\delta_z+\vartheta_f\delta_f\right), b_{31}=\frac{\Pi_m\mu_h}{\Pi_h\mu_m}\left(\vartheta_m\delta_m+\vartheta_z\delta_z+\vartheta_f\delta_f\right), b_{32}=-\mu_m, b_{33}=0, b_{34}=0, b_{35}=0$$

The stability of the multi-infections model around$\mathcal{E}_{mfz}$ in equation (17) is established if all the eigenvalues of the Jacobian have a negative real part. Since the first, ninth, tenth, eleventh and twelfth columns have only diagonal entries, it is obvious that five of the eigenvalues thus $-\mu_h, -(\mu_h+\psi), -(\mu_h+\varphi), -\mu_h$ and $-\mu_m$ have negative real parts. Hence the stability of the disease free equilibrium is dependant on the eigenvalues of the sub-matrix of the Jacobian matrix.

$$J-I\lambda=\begin{vmatrix} -a_1 & 0 & 0 & a_{17} & a_{16} & 0 & 0 & a_2 & 0 & 0 \\ 0 & -a_5 & 0 & 0 & a_{15} & a_{17} & 0 & 0 & a_3 & 0 \\ 0 & 0 & -a_6 & a_{15} & a_{16} & 0 & 0 & 0 & 0 & a_4 \\ 0 & 0 & 0 & -a_7 & 0 & 0 & a_{16} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_8 & 0 & a_{17} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -a_9 & a_{15} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -a_{10} & 0 & 0 & 0 \\ a_{11} & 0 & 0 & a_{11} & a_{11} & 0 & a_{11} & -a_{12} & 0 & 0 \\ 0 & a_{13} & 0 & 0 & a_{13} & a_{13} & a_{13} & 0 & -a_{12} & 0 \\ 0 & 0 & a_{14} & a_{14} & a_{14} & 0 & a_{14} & 0 & 0 & -a_{12} \end{vmatrix}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (18)$$

$$a_1=\left(\mu_h+\tau_m+\eta+\lambda\right), a_2=\vartheta_m\delta_m, a_3=\vartheta_f\delta_f, a_4=\vartheta_z\delta_z, a_5=\left(\mu_h+\tau_f+\lambda\right), a_6=\left(\mu_h+\tau_z+\lambda\right),$$
$$a_7=\left(\mu_h+\tau_m+\tau_z+\eta+\lambda\right), a_8=\left(\mu_h+\tau_m+\tau_f+\eta+\lambda\right), a_9=\left(\mu_h+\tau_f+\tau_z+\lambda\right),$$
$$a_{10}=\left(\mu_h+\tau_m+\tau_f+\tau_z+\eta+\lambda\right), a_{11}=\frac{\vartheta_m\delta_m\Pi_m\mu_h}{\Pi_h\mu_m}, a_{12}=(\mu_m+\lambda), a_{13}=\frac{\vartheta_f\delta_f\Pi_m\mu_h}{\Pi_h\mu_m}, a_{14}=\frac{\vartheta_z\delta_z\Pi_m\mu_h}{\Pi_h\mu_m}$$
$$a_{15}=\tau_m, a_{16}=\tau_f, a_{17}=\tau_z$$

From equation (18) we obtain the following sub-matrices.

$$\begin{bmatrix} -a_5 & a_3 \\ a_{13} & -a_{12} \end{bmatrix}=\begin{bmatrix} -\left(\mu_h+\tau_f+\lambda\right) & \vartheta_f\delta_f \\ \frac{\vartheta_f\delta_f\Pi_m\mu_h}{\Pi_h\mu_m} & -(\mu_m+\lambda) \end{bmatrix} \quad (19)$$

$$\begin{bmatrix} -a_1 & a_2 \\ a_{11} & -a_{12} \end{bmatrix}=\begin{bmatrix} -(\mu_h+\tau_m+\eta+\lambda) & \vartheta_m\delta_m \\ \frac{\vartheta_m\delta_m\Pi_m\mu_h}{\Pi_h\mu_m} & -(\mu_m+\lambda) \end{bmatrix} \quad (20)$$

$$\begin{bmatrix} -a_6 & a_4 \\ a_{14} & -a_{12} \end{bmatrix}=\begin{bmatrix} -(\mu_h+\tau_z+\lambda) & \vartheta_z\delta_z \\ \frac{\vartheta_z\delta_z\Pi_m\mu_h}{\Pi_h\mu_m} & -(\mu_m+\lambda) \end{bmatrix} \quad (21)$$

From equations (19), (20) and (21), we obtain the following characteristic polynomial;

$$x^2+\left(2\lambda+\mu_h+\mu_m+\tau_f\right)x-\frac{\vartheta_f^2\delta_f^2\Pi_m\mu_h}{\Pi_h\mu_m}+\lambda^2+\left(\mu_h+\mu_m+\tau_f\right)\lambda+\tau_f\mu_m+\mu_h\mu_m=0 \quad (22)$$

$$x^2 + (2\lambda + \mu_h + \mu_m + \eta + \tau_m)x - \frac{\vartheta_m^2 \delta_m^2 \Pi_m \mu_h}{\Pi_h \mu_m} + \lambda^2 + (\mu_h + \mu_m + \tau_m)\lambda + \lambda\eta + \tau_m\mu_m + \mu_h\mu_m = 0 \quad (23)$$

$$x^2 + (2\lambda + \mu_h + \mu_m + \tau_z)x - \frac{\vartheta_z^2 \delta_z^2 \Pi_m \mu_h}{\Pi_h \mu_m} + \lambda^2 + (\mu_h + \mu_m + \tau_z)\lambda + \mu_h\mu_m + \tau_z\mu_m = 0 \quad (24)$$

By applying Routh-Hurwitz criteria for dimension $n = 2$ requires that given a polynomial of the form,

$x^2 + a_1 x + a_2 = 0$

The coefficient of a_1 and a_2 be greater than zero. Hence applying the same principle, equation (22) can be rewritten in the form

$$x^2 + f_1 x + f_2 = 0 \ldots\ldots\ldots \quad (25)$$

Where

$$f_1 = 2\lambda + \mu_h + \mu_m + \tau_f \quad (26)$$

$$f_2 = -\frac{\vartheta_f^2 \delta_f^2 \Pi_m \mu_h}{\Pi_h \mu_m} + \lambda^2 + \lambda\mu_h + \lambda\mu_m + \lambda\tau_f + \mu_h\mu_m + \tau_f\mu_m \quad (27)$$

Since $f_1 > 0$, the criteria requires that $f_2 > 0$. Hence to achieve that we perform some algebraic manipulation to obtain.

$$f_2 = -\frac{(\mu_h + \tau_f)\Pi_h \mu_m^2 R_f}{\Pi_h \mu_m} + \lambda(\lambda + \mu_h + \mu_m + \tau_f) + \mu_m(\mu_h + \tau_f)$$
$$= -R_f\mu_m(\mu_h + \tau_f) + \mu_m(\mu_h + \tau_f) + \lambda(\lambda + \mu_h + \mu_m + \tau_f)$$
$$= \lambda(\lambda + \mu_h + \mu_m + \tau_f) + \mu_m(\mu_h + \tau_f)[1 - R_f]$$

It is observed that for f_2 to be greater than zero requires$R_f < 1$.

Similarly, equation (23) is written in the form

$x^2 + m_1 x + m_2 = 0$

Which implies

$$m_1 = 2\lambda + \mu_h + \mu_m + \eta + \tau_m,$$
$$m_2 = -\frac{\vartheta_m^2 \delta_m^2 \Pi_m \mu_h}{\Pi_h \mu_m} + \lambda^2 + (\mu_h + \mu_m + \tau_m + \eta)\lambda + (\tau_m + \mu_h)\mu_m \ldots\ldots\ldots\ldots \quad (28)$$

Since $m_1 > 0$, the criteria requires that $m_2 > 0$. Hence to achieve that we perform some algebraic manipulation to obtain.

$$m_2 = -\frac{(\mu_h + \eta + \tau_m)\Pi_h \mu_m^2 R_m}{\Pi_h \mu_m} + \lambda(\lambda + \mu_h + \eta + \mu_m + \tau_m) + \mu_m(\mu_h + \tau_m + \eta)$$
$$= \lambda(\lambda + \mu_h + \eta + \mu_m + \tau_m) + \mu_m(\mu_h + \tau_m + \eta)[1 - R_m]$$

In a similar manner, equation (24) can be rewritten in the form

$x^2 + z_1 x + z_2 = 0$

Implies

$$z_1 = 2\lambda + \mu_h + \mu_m + \eta + \tau_z \quad (29)$$

$$z_2 = -\frac{\vartheta_z^2 \delta_z^2 \Pi_m \mu_h}{\Pi_h \mu_m} + \lambda^2 + \lambda\mu_h + \lambda\mu_m + \lambda\tau_z + \tau_z\mu_m + \mu_h\mu_m \quad (30)$$

In order to makez_2 greater than zero,

$$z_2 = -(\mu_h + \tau_z)\Pi_h\mu_m R_m + \lambda(\lambda + \mu_h + \mu_m + \tau_z) + \mu_m(\mu_h + \tau_z)$$
$$= \lambda(\lambda + \mu_h + \mu_m + \tau_z) + \mu_m(\mu_h + \tau_z)[1 - R_z]$$

It is observed that for, $f_2, m_2 and z_2$ to be positive in order to satisfy Routh-Huriwitz criteria requires that $R_f, R_m and R_z$be less than 1 the condition that the various reproduction numbers be less than one indicates that the disease free equilibrium is locally asymptotically stable.

3.1.3 Global stability at the disease free equilibrium for multi-infection model (Malaria-Zika-Elephantiasis):

In this section, the global stability of the disease free equilibrium is proved as stated in the theorem below.

Theorem 3

The DFE$(\mathcal{E}_{mzf})$ of system of equation (1) is globally asymptotically stable if $R_{mzf} < 1$ and unstable if $R_{mzf} > 1$

Proof: We rewrite the model as

$$\left.\begin{aligned}\frac{dF}{dt} &= \chi(F,G)\\ \frac{dG}{dt} &= \gamma(F,G)\end{aligned}\right\} \dots\dots\dots\dots \quad (31)$$

$F = I_m, I_f, I_z, I_{mz}, I_{mf}, I_{zf}, I_{mzf}, I_p, I_w, I_a$ represents infectious class and un-infectious class as $G = S_h, R_m, R_f, R_z, S_m$. We define the two valued functions as $\chi(F,G)$ with $F \in \mathbb{R}_+^{10}$ and $\gamma(F,G)$ with $G \in \mathbb{R}_+^5$and are given by

$$\chi(F,G) = \left\{\begin{array}{c}
\frac{\vartheta_m\delta_m I_p}{N_h}S_h - \tau_m I_m - (\mu_h + \eta)I_m + \tau_f I_{mf} - \frac{\vartheta_f\delta_f I_w}{N_h}I_m + \tau_z I_{mz} - \frac{\vartheta_z\delta_z I_a}{N_h}I_m \\
\frac{\vartheta_f\delta_f I_w}{N_h}S_h - (\mu_h + \tau_f)I_f - \frac{\vartheta_m\delta_m I_p}{N_h}I_f + \tau_m I_{mf} + \tau_z I_{zf} - \frac{\vartheta_z\delta_z I_a}{N_h}I_f \\
\frac{\vartheta_z\delta_z I_a}{N_h}S_h - (\tau_z + \mu_h)I_z - \frac{\vartheta_m\delta_m I_p}{N_h}I_z + \tau_m I_{mz} + \tau_f I_{zf} - \frac{\vartheta_f\delta_f I_w}{N_h}I_z \\
\frac{\vartheta_z\delta_z I_a}{N_h}I_m - (\mu_h + \eta + \tau_m + \tau_z)I_{mz} + \tau_f I_{mzf} - \frac{\vartheta_f\delta_f I_w}{N_h}I_{mz} + \frac{\vartheta_m\delta_m I_p}{N_h}I_z \\
\frac{\vartheta_f\delta_f I_w}{N_h}I_m - (\mu_h + \eta + \tau_m + \tau_f)I_{mf} + \tau_z I_{mzf} - \frac{\vartheta_z\delta_z I_a}{N_h}I_{mf} + \frac{\vartheta_m\delta_m I_p}{N_h}I_f \\
\frac{\vartheta_z\delta_z I_a}{N_h}I_f - (\mu_h + \tau_f + \tau_z)I_{zf} + \tau_m I_{mzf} - \frac{\vartheta_m\delta_m I_p}{N_h}I_{zf} + \frac{\vartheta_f\delta_f I_w}{N_h}I_z \\
\frac{\vartheta_m\delta_m I_p}{N_h}I_{zf} - (\mu_h + \eta + \tau_m + \tau_f + \tau_z)I_{mzf} + \frac{\vartheta_f\delta_f I_w}{N_h}I_{mz} + \frac{\vartheta_z\delta_z I_a}{N_h}I_{mf} \\
\frac{\vartheta_m\delta_m(I_m + I_{mzf})}{N_h}S_m - \mu_m I_p, \frac{\vartheta_f\delta_f(I_f + I_{mzf})}{N_h}S_m - \mu_m I_w, \frac{\vartheta_z\delta_z(I_z + I_{mzf})}{N_h}S_m - \mu_m I_a \\
and
\end{array}\right\} \dots . (32)$$

$$\gamma(F,G)=\begin{Bmatrix}\tau_m I_m-(\mu_h+\psi)R_m\\ \tau_f I_f-(\mu_h+\varphi)R_f\\ \tau_z I_z-\mu_h R_z\\ \Pi_h+\psi R_m+\varphi R_f-\frac{\vartheta_m\delta_m I_p}{N_h}S_h-\mu_h S_h-\frac{\vartheta_z\delta_z I_a}{N_h}S_h-\frac{\vartheta_f\delta_f I_w}{N_h}S_h\\ \Pi_m-\mu_m S_m-\frac{\vartheta_m\delta_m\left(I_m+I_{mz}+I_{mf}+I_{mzf}\right)}{N_h}S_m\\ -\frac{\vartheta_f\delta_f\left(I_f+I_{mf}+I_{zf}+I_{mzf}\right)}{N_h}S_m-\frac{\vartheta_z\delta_z\left(I_z+I_{mz}+I_{zf}+I_{mzf}\right)}{N_h}S_m\end{Bmatrix} \quad (33)$$

Now the reduced form of the system:

$$\frac{dG}{dt}=\gamma(0,G)$$

$$\left.\begin{aligned}\frac{dS_h}{dt}&=\Pi_h+\psi R_m+\varphi R_f-\frac{\vartheta_m\delta_m I_p}{N_h}S_h-\mu_h S_h-\frac{\vartheta_z\delta_z I_a}{N_h}S_h-\frac{\vartheta_f\delta_f I_w}{N_h}S_h\\ \frac{dR_m}{dt}&=\tau_m I_m-(\mu_h+\psi)R_m\\ \frac{dR_f}{dt}&=\tau_f I_f-(\mu_h+\varphi)R_f\\ \frac{dR_z}{dt}&=\tau_z I_z-\mu_h R_z\\ \frac{dS_m}{dt}&=\Pi_m-\mu_m S_m-\frac{\vartheta_m\delta_m\left(I_m+I_{mz}+I_{mf}+I_{mzf}\right)}{N_h}S_m\\ &-\frac{\vartheta_f\delta_f\left(I_f+I_{mf}+I_{zf}+I_{mzf}\right)}{N_h}S_m-\frac{\vartheta_z\delta_z\left(I_z+I_{mz}+I_{zf}+I_{mzf}\right)}{N_h}S_m\end{aligned}\right\} \quad (34)$$

$F^*=\left(S_h^*,R_m^*,R_f^*,R_z^*,S_m^*\right)=\left(\frac{\Pi_h}{\mu_h}0,0,0,\frac{\Pi_m}{\mu_m}\right)$ is globally asymptotically stable equilibrium point for the reduced form of the system$\frac{dG}{dt}=\gamma(0,G)$. Therefore $R_m(t)=R_m(0)e^{-(\mu_h+\psi)t}$ turns to zero as $t\longrightarrow\infty$ and $R_f(t)=R_f(0)e^{-(\mu_h+\varphi)t}$ also turns to zero as $t\longrightarrow\infty$. In $S_h(t)=\Pi_h+\psi\{R_m(0)e^{-(\mu_h+\psi)t}\}+\varphi\{R_f(0)e^{-(\mu_h+\varphi)t}\}-\mu_h\left\{\frac{\Pi_h}{\mu_h}+\left[S_h(0)-\frac{\Pi_h}{\mu_h}\right]\right\}e^{-\mu_h t}\longrightarrow\frac{\Pi_h}{\mu_h}$ as $t\longrightarrow\infty$. This asymptotic dynamics is independent of intital conditions in Ω. Hence the convergence of the solution (34) is global in Ω. Truly $\chi(F,G)$ satisfies the following two conditions given as H_2 in [9] namely

1.$\chi(0,G)=0$ and

$$2.\chi(F,G)=TG-\bar{\chi}(F,G),\bar{\chi}(F,G)\geq 0\text{on}\Omega \quad (35)$$

$$T=D_Y\chi\left(0,G\right)=\begin{vmatrix}-z_1&0&0&\tau_z&\tau_f&0&0&z_2&0&0\\0&-z_3&0&0&\tau_m&\tau_z&0&0&z_4&0\\0&0&-z_5&\tau_m&0&0&\tau_f&0&0&z_6\\0&0&0&-z_7&0&0&\tau_f&0&0&0\\0&0&0&0&-z_8&0&\tau_z&0&0&0\\0&0&0&0&0&-z_9&\tau_m&-0&0&0\\0&0&0&z_{10}&z_{11}&0&-z_{12}&0&0&0\\z_{13}&0&0&0&0&0&z_{13}&-\mu_m&0&0\\0&z_{14}&0&0&0&0&z_{14}&0&-\mu_m&0\\0&0&z_{15}&0&0&0&z_{15}&0&0&-\mu_m\end{vmatrix} \quad \text{.....(36)}$$

$$z_1 = \left(\mu_h + \tau_m + \eta\right), z_2 = \frac{\vartheta_m \delta_m S_h}{N_h}, z_3 = \left(\mu_h + \tau_f\right), z_4 = \frac{\vartheta_f \delta_f S_h}{N_h}, z_5 = \left(\mu_h + \tau_z\right), z_6 = \frac{\vartheta_z \delta_z S_h}{N_h},$$

$$z_7 = \left(\mu_h + \eta + \tau_m + \tau_z\right) z_8 = \left(\mu_h + \eta + \tau_m + \tau_f\right), z_9 = \left(\mu_h + \tau_z + \tau_f\right), z_{10} = \frac{\vartheta_f \delta_f I_w}{N_h}, z_{11} = \frac{\vartheta_z \delta_z I_a}{N_h},$$

$$z_{12} = \left(\mu_h + \eta + \tau_m + \tau_f + \tau_z\right) z_{13} = \frac{\vartheta_m \delta_m \Pi_m \mu_h}{\Pi_h \mu_m}, z_{14} = \frac{\vartheta_f \delta_f \Pi_m \mu_h}{\Pi_h \mu_m}, z_{15} = \frac{\vartheta_z \delta_z \Pi_m \mu_h}{\Pi_h \mu_m}$$

$$\dot{\chi}_i\left(F, G\right) = \begin{pmatrix} \alpha_m\left(1 - \frac{S_h}{N_h}\right) + \left(\alpha_f + \alpha_z\right) I_m \\ \alpha_f\left(1 - \frac{S_h}{N_h}\right) + \left(\alpha_m + \alpha_z\right) I_f \\ \alpha_z\left(1 - \frac{S_h}{N_h}\right) + \left(\alpha_m + \alpha_f\right) I_z \\ -\frac{\alpha_z \delta_z I_a I_m}{N_h} + \frac{\alpha_f \delta_f I_w I_{mz}}{N_h} - \frac{\alpha_m \delta_m I_p I_z}{N_h} \\ -\frac{\alpha_f \delta_f I_w I_m}{N_h} + \frac{\alpha_z \delta_z I_a I_{mf}}{N_h} - \frac{\alpha_m \delta_m I_p I_f}{N_h} \\ -\frac{\alpha_z \delta_z I_a I_f}{N_h} + \frac{\alpha_m \delta_m I_p I_{zf}}{N_h} - \frac{\alpha_f \delta_f I_w I_z}{N_h} \\ -\frac{\alpha_m \delta_m I_p I_{zf}}{N_h} - \frac{\alpha_f \delta_f I_w I_{mz}}{N_h} - \frac{\alpha_z \delta_z I_a I_{mf}}{N_h} \\ \alpha_m \delta_m \left(I_m + I_{mzf}\right)\left(1 - \frac{S_h}{N_h}\right) \\ \alpha_f \delta_f \left(I_f + I_{mzf}\right)\left(1 - \frac{S_h}{N_h}\right) \\ \alpha_z \delta_z \left(I_z + I_{mzf}\right)\left(1 - \frac{S_h}{N_h}\right) \end{pmatrix} where\ i = 1, 2, 3 \ldots\ldots 10. \quad . (37)$$

It is shown that$\dot{\chi_4}(F, G) < 0, \dot{\chi_5}(F, G) < 0, \dot{\chi_6}(F, G) < 0\ and \dot{\chi_7}(F, G) < 0$ and so the conditions in (35) are not met. Hence the DFE ε_{mzf}may not be globally asymptotically stable if$R_{mzf} < 1$.

3.1.4 Bifurcation

To determine the endemic equilibrium of the system (1) involves tedious computation. And as result of this, the Center manifold theorem as used [10] is applied in this situation. The system of equation (1) can be rewritten in a dimensionless state variables of the Multi-infection model as follows:

$$x_1 = S_h, x_2 = I_m, x_3 = I_f, x_4 = I_z, x_5 = I_{mz}, x_6 = I_{mf}, x_7 = I_{fz}, x_8 = I_{mfz}, x_9 = R_m,$$

$$x_{10} = R_f, x_{11} = R_z, x_{12} = S_m, x_{13} = I_p, x_{14} = I_w, x_{15} = I_a$$

The system (1) can be written in a vector form as

$$\frac{dX_i}{dt} = F(X_i)$$

Here, $X_i = (x_1, x_2, x_3, , , , , x_{15})^T, F = (f_1, f_2, f_3, , , , , f_{15})^T$

The system (1) is now as follows

$$\left.\begin{aligned}
\frac{dx_1}{dt} &= \Pi_h + \psi x_9 + \varphi x_{10} - \frac{\vartheta_m \delta_m x_{13}}{N_h} x_1 - \mu_h x_1 - \frac{\vartheta_z \delta_z x_{15}}{N_h} x_1 - \frac{\vartheta_f \delta_f x_{14}}{N_h} x_1 = f_1 \\
\frac{dx_2}{dt} &= \frac{\vartheta_m \delta_m x_{13}}{N_h} x_1 - (\mu_h + \eta + \tau_m) x_2 + \tau_f x_6 - \frac{\vartheta_f \delta_f x_{14}}{N_h} x_2 + \tau_z x_5 - \frac{\vartheta_z \delta_z x_{15}}{N_h} x_6 = f_2 \\
\frac{dx_3}{dt} &= \frac{\vartheta_f \delta_f x_{14}}{N_h} x_1 - \mu_h x_3 - \frac{\vartheta_m \delta_m x_{13}}{N_h} x_3 + \tau_m x_6 - \tau_f x_3 + \frac{\vartheta_z \delta_z x_{15}}{N_h} x_7 - \frac{\vartheta_z \delta_z x_{15}}{N_h} x_3 = f_3 \\
\frac{dx_4}{dt} &= \frac{\vartheta_z \delta_z x_{15}}{N_h} x_1 - \tau_z x_4 - \frac{\vartheta_m \delta_m x_{13}}{N_h} x_4 + \tau_m x_5 + \tau_f x_7 - \frac{\vartheta_f \delta_f x_{14}}{N_h} x_4 - \mu_h x_4 = f_4 \\
\frac{dx_5}{dt} &= \frac{\vartheta_z \delta_z x_{15}}{N_h} x_5 - \tau_z x_5 - (\mu_h + \eta) x_5 + \tau_f x_8 - \frac{\vartheta_f \delta_f x_{14}}{N_h} x_5 + \frac{\vartheta_m \delta_m x_{13}}{N_h} x_4 - \tau_m x_5 = f_5 \\
\frac{dx_6}{dt} &= \frac{\vartheta_f \delta_f x_{14}}{N_h} x_2 - \tau_f x_6 - (\mu_h + \eta) x_6 + \tau_z x_8 - \frac{\vartheta_z \delta_z x_{15}}{N_h} x_6 + \frac{\vartheta_m \delta_m x_{13}}{N_h} x_3 - \tau_m x_6 = f_6 \\
\frac{dx_7}{dt} &= \frac{\vartheta_z \delta_z x_{15}}{N_h} x_3 - \tau_f x_7 - \mu_h x_7 + \tau_m x_8 - \frac{\vartheta_m \delta_m x_{13}}{N_h} x_7 + \frac{\vartheta_f \delta_f x_{14}}{N_h} x_4 - \tau_z x_7 = f_7 \\
\frac{dx_8}{dt} &= \frac{\vartheta_m \delta_m x_{13}}{N_h} x_7 - \tau_m x_8 - (\mu_h + \eta) x_8 - \tau_f x_8 + \frac{\vartheta_f \delta_f x_{14}}{N_h} x_5 + \frac{\vartheta_z \delta_z x_{15}}{N_h} x_6 - \tau_z x_8 = f_8 \\
\frac{dx_9}{dt} &= \tau_m x_6 - (\mu_h + \psi) x_9 = f_9 \\
\frac{dx_{10}}{dt} &= \tau_f x_3 - (\mu_h + \varphi) x_{10} = f_{10} \\
\frac{dx_{11}}{dt} &= \tau_z x_4 - \mu_h x_{11} = f_{11} \\
\frac{dx_{12}}{dt} &= \Pi_m - \mu_m x_{12} - \frac{\vartheta_m \delta_m (x_6 + x_5 + x_6 + x_8)}{N_h} x_{12} - \frac{\vartheta_f \delta_f (x_3 + x_6 + x_7 + x_8)}{N_h} x_{12} \\
&\quad - \frac{\vartheta_z \delta_z (x_4 + x_5 + x_7 + x_8)}{N_h} x_{12} = f_{12} \\
\frac{dx_{13}}{dt} &= \frac{\vartheta_m \delta_m (x_2 + x_5 + x_6 + x_8)}{N_h} x_{12} - \mu_m x_{13} = f_{13} \\
\frac{dx_{14}}{dt} &= \frac{\vartheta_f \delta_f (x_3 + x_6 + x_7 + x_8)}{N_h} x_{12} - \mu_m x_{14} = f_{14} \\
\frac{dx_{15}}{dt} &= \frac{\vartheta_z \delta_z (x_4 + x_5 + x_7 + x_8)}{N_h} x_{12} - \mu_m x_{15} = f_{15}
\end{aligned}\right\} \ldots\ldots (38)$$

$N_h = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11}$ and $N_m = x_{12} + x_{13} + x_{14} + x_{15}$

With bifurcation parameter $\beta^* = \vartheta_m^2$. Here, consider a situation when $R_{mzf} = 1$ and assuming that R_{ma} is greater than both R_z and R_f, then solving for β^* at $R_{mzf} = R_{ma} = 1$ gives

$$\beta^* = \frac{(\mu_h+\eta+\tau_m)\Pi_h\mu_m^2}{\delta_m^2\Pi_m\mu_h} \tag{39}$$

Here, the method involves evaluation of Jacobian matrix at the system (1) at the disease free equilibrium denoted by$\mathcal{E}_{mfz}$. This becomes

$$J = \begin{pmatrix}
-b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \psi & \varphi & 0 & 0 & -b_2 & -b_3 & -b_4 \\
0 & -b_6 & 0 & 0 & \tau_z & \tau_f & 0 & 0 & 0 & 0 & 0 & 0 & b_2 & 0 & 0 \\
0 & 0 & -b_9 & 0 & 0 & \tau_m & \tau_z & 0 & 0 & 0 & 0 & 0 & 0 & b_3 & 0 \\
0 & 0 & 0 & -b_{13} & \tau_m & 0 & \tau_f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_4 \\
0 & 0 & 0 & 0 & -b_{15} & 0 & 0 & \tau_f & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -b_{18} & 0 & \tau_z & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -b_{20} & \tau_m & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -b_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \tau_m & 0 & 0 & 0 & 0 & 0 & 0 & -b_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \tau_f & 0 & 0 & 0 & 0 & 0 & 0 & -b_{24} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \tau_z & 0 & 0 & 0 & 0 & 0 & 0 & -\mu_h & 0 & 0 & 0 & 0 \\
0 & -b_{25} & -b_{26} & -b_{27} & -b_{28} & -b_{29} & -b_{30} & -b_{31} & 0 & 0 & 0 & -b_{32} & 0 & 0 & 0 \\
0 & b_{25} & 0 & 0 & b_{25} & 0 & b_{25} & b_{25} & 0 & 0 & 0 & b_{33} & -\mu_m & 0 & 0 \\
0 & 0 & b_{26} & 0 & 0 & b_{26} & b_{26} & b_{26} & 0 & 0 & 0 & b_{34} & 0 & -\mu_m & 0 \\
0 & 0 & 0 & b_{27} & b_{27} & b_{27} & 0 & b_{27} & 0 & 0 & 0 & b_{35} & 0 & 0 & -\mu_m
\end{pmatrix} \tag{40}$$

$$b_1 = \mu_h, b_2 = \vartheta_m\delta_m, b_3 = \vartheta_f\delta_f, b_4 = \vartheta_z\delta_z, b_5 = 0, b_6 = \left(\mu_h+\eta+\tau_m\right), b_7 = 0, b_8 = 0, b_9 = \left(\mu_h+\tau_f\right)$$

$$, b_{10} = 0, b_{11} = 0, b_{12} = 0, b_{13} = \left(\mu_h+\tau_z\right), b_{14} = 0, b_{15} = \left(\mu_h+\eta+\tau_m+\tau_z\right), b_{16} = 0, b_{17} = 0, b_{18} = \left(\mu_h+\eta+\tau_m+\tau_f\right),$$

$$b_{19} = 0, b_{20} = \left(\mu_h+\tau_f+\tau_z\right), b_{21} = 0, b_{22} = \left(\mu_h+\eta+\tau_m+\tau_f+\tau_z\right), b_{23} = \left(\mu_h+\psi\right), b_{24} = \left(\mu_h+\varphi\right),$$

$$b_{25} = \frac{\vartheta_m\delta_m\Pi_m\mu_h}{\Pi_h\mu_m}, b_{26} = \frac{\vartheta_f\delta_f\Pi_m\mu_h}{\Pi_h\mu_m}, b_{27} = \frac{\vartheta_z\delta_z\Pi_m\mu_h}{\Pi_h\mu_m}, b_{28} = \frac{\Pi_m\mu_h}{\Pi_h\mu_m}\left(\vartheta_m\delta_m+\vartheta_z\delta_z\right), b_{29} = \frac{\Pi_m\mu_h}{\Pi_h\mu_m}\left(\vartheta_m\delta_m+\vartheta_f\delta_f\right),$$

$$b_{30} = \frac{\Pi_m\mu_h}{\Pi_h\mu_m}\left(\vartheta_z\delta_z+\vartheta_f\delta_f\right), b_{31} = \frac{\Pi_m\mu_h}{\Pi_h\mu_m}\left(\vartheta_m\delta_m+\vartheta_z\delta_z+\vartheta_f\delta_f\right), b_{32} = -\mu_m, b_{33} = 0, b_{34} = 0, b_{35} = 0$$

The right eigenvector associated with the eigenvalues are given below

$$J = \begin{pmatrix}
-b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \psi & \varphi & 0 & 0 & -b_2 & -b_3 & -b_4 \\
0 & -b_6 & 0 & 0 & \tau_z & \tau_f & 0 & 0 & 0 & 0 & 0 & 0 & b_2 & 0 & 0 \\
0 & 0 & -b_9 & 0 & 0 & \tau_m & \tau_z & 0 & 0 & 0 & 0 & 0 & 0 & b_3 & 0 \\
0 & 0 & 0 & -b_{13} & \tau_m & 0 & \tau_f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_4 \\
0 & 0 & 0 & 0 & -b_{15} & 0 & 0 & \tau_f & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -b_{18} & 0 & \tau_z & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -b_{20} & \tau_m & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -b_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \tau_m & 0 & 0 & 0 & 0 & 0 & 0 & -b_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \tau_f & 0 & 0 & 0 & 0 & 0 & 0 & -b_{24} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \tau_z & 0 & 0 & 0 & 0 & 0 & 0 & -\mu_h & 0 & 0 & 0 & 0 \\
0 & -b_{25} & -b_{26} & -b_{27} & -b_{28} & -b_{29} & -b_{30} & -b_{31} & 0 & 0 & 0 & -b_{32} & 0 & 0 & 0 \\
0 & b_{25} & 0 & 0 & b_{25} & 0 & b_{25} & b_{25} & 0 & 0 & 0 & b_{33} & -\mu_m & 0 & 0 \\
0 & 0 & b_{26} & 0 & 0 & b_{26} & b_{26} & b_{26} & 0 & 0 & 0 & b_{34} & 0 & -\mu_m & 0 \\
0 & 0 & 0 & b_{27} & b_{27} & b_{27} & 0 & b_{27} & 0 & 0 & 0 & b_{35} & 0 & 0 & -\mu_m
\end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \\ w_9 \\ w_{10} \\ w_{11} \\ w_{12} \\ w_{13} \\ w_4 \\ w_{15} \end{pmatrix} \tag{41}$$

$$\left.\begin{aligned}
w_1 &= \frac{\psi w_9 + \varphi w_{10} - \vartheta_m \delta_m w_{13} - \vartheta_f \delta_f w_{14} - \vartheta_z \delta_z w_{15}}{\mu_h} \\
w_2 &= \frac{\tau_z w_5 + \tau_f w_6 + \vartheta_m \delta_m w_{13}}{(\mu_h + \eta + \tau_m)} \\
w_3 &= \frac{\tau_m w_6 + \tau_z w_7 + \vartheta_f \delta_f w_{14}}{(\mu_h + \tau_f)} \\
w_4 &= \frac{\tau_m w_5 + \tau_f w_7 + \vartheta_z \delta_z w_{15}}{(\mu_h + \tau_z)} \\
w_5 &= 0 \\
w_6 &= 0 \\
w_7 &= 0 \\
w_8 &= 0 \\
w_9 &= \frac{\tau_m w_2}{(\mu_h + \psi)} \\
w_{10} &= \frac{\tau_f w_3}{(\mu_h + \varphi)} \\
w_{11} &= \frac{\tau_z w_4}{\mu_h} \\
w_{12} &= \frac{\Pi_m \mu_h (\vartheta_m \delta_m w_2 + \vartheta_f \delta_f w_3 + \vartheta_z \delta_z w_4)}{\Pi_h \mu_m^2} \\
w_{13} &= \frac{\vartheta_m \delta_m \Pi_m \mu_h w_2}{\Pi_h \mu_m^2} \\
w_{14} &= \frac{\vartheta_f \delta_f \Pi_m \mu_h w_3}{\Pi_h \mu_m^2} \\
w_{15} &= \frac{\vartheta_z \delta_z \Pi_m \mu_h w_4}{\Pi_h \mu_m^2}
\end{aligned}\right\} \quad (42)$$

The left eigenvector is the transpose of system (41) and this is also evaluated as follows

$$J = \begin{pmatrix}
-b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -b_6 & 0 & 0 & 0 & 0 & 0 & 0 & \tau_m & 0 & 0 & -b_{25} & b_{25} & 0 & 0 \\
0 & 0 & -b_9 & 0 & 0 & 0 & 0 & 0 & 0 & \tau_f & 0 & -b_{26} & 0 & b_{26} & 0 \\
0 & 0 & 0 & -b_{13} & 0 & 0 & 0 & 0 & 0 & 0 & \tau_z & -b_{27} & 0 & 0 & b_{27} \\
0 & \tau_z & 0 & \tau_m & -b_{15} & 0 & 0 & 0 & 0 & 0 & 0 & -b_{28} & b_{25} & 0 & b_{27} \\
0 & \tau_f & \tau_m & 0 & 0 & -b_{18} & 0 & 0 & 0 & 0 & 0 & -b_{29} & 0 & b_{26} & b_{27} \\
0 & 0 & \tau_z & \tau_f & 0 & 0 & -b_{20} & 0 & 0 & 0 & 0 & -b_{30} & b_{25} & b_{26} & 0 \\
0 & 0 & 0 & 0 & \tau_f & \tau_z & \tau_m & -b_{22} & 0 & 0 & 0 & -b_{31} & b_{25} & b_{26} & b_{27} \\
\psi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -b_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\
\varphi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -b_{24} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu_h & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -b_{32} & b_{33} & b_{34} & b_{35} \\
-b_2 & b_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu_m & 0 & 0 \\
-b_3 & 0 & b_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu_m & 0 \\
-b_4 & 0 & 0 & b_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu_m
\end{pmatrix}
\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \\ v_{11} \\ v_{12} \\ v_{13} \\ v_{14} \\ v_{15} \end{pmatrix} \quad (43)$$

Solving for the left eigenvector gives

$$\left.\begin{aligned} v_1 &= 0 \\ v_2 &= \frac{\vartheta_m \delta_m \Pi_m \mu_h v_{13}}{(\mu_h + \eta + \tau_m)\Pi_h \mu_m} \\ v_3 &= \frac{\vartheta_f \delta_f \Pi_m \mu_h v_{14}}{(\mu_h + \tau_f)\Pi_h \mu_m} \\ v_4 &= \frac{\vartheta_z \delta_z \Pi_m \mu_h v_{15}}{(\mu_h + \tau_z)\Pi_h \mu_m} \\ v_5 &= \frac{\tau_z v_2 + \tau_m v_4 + \vartheta_m \delta_m \Pi_m \mu_h v_{13} + \vartheta_z \delta_z \Pi_m \mu_h v_{15}}{(\mu_h + \eta + \tau_m + \tau_z)\Pi_h \mu_m} \\ v_6 &= \frac{\tau_f v_2 + \tau_m v_3 + \vartheta_f \delta_f \Pi_m \mu_h v_{14} + \vartheta_z \delta_z \Pi_m \mu_h v_{15}}{(\mu_h + \eta + \tau_m + \tau_f)\Pi_h \mu_m} \\ v_7 &= \frac{\tau_z v_3 + \tau_f v_4 + \vartheta_m \delta_m \Pi_m \mu_h v_{13} + \vartheta_f \delta_f \Pi_m \mu_h v_{14}}{(\mu_h + \tau_f + \tau_z)\Pi_h \mu_m} \\ v_8 &= \frac{\tau_f v_5 + \tau_z v_6 + \tau_m v_7 + \vartheta_m \delta_m \Pi_m \mu_h v_{13} + \vartheta_f \delta_f \Pi_m \mu_h v_{14} + \vartheta_z \delta_z \Pi_m \mu_h v_{15}}{(\mu_h + \eta + \tau_m + \tau_f + \tau_z)\Pi_h \mu_m} \\ v_9 &= 0 \\ v_{10} &= 0 \\ v_{11} &= 0 \\ v_{12} &= 0 \\ v_{13} &= \frac{\vartheta_m \delta_m v_2}{\mu_m} \\ v_{14} &= \frac{\vartheta_f \delta_f v_3}{\mu_m} \\ v_{15} &= \frac{\vartheta_z \delta_z v_4}{\mu_m} \end{aligned}\right\} \ldots\ldots (44)$$

After going through a lot of computation we arrive at

Thus the expression for a is g**iven as**

$$a = v_2 w_1 w_{13}\left[\frac{\partial^2 f_2}{\partial x_1 \partial x_{13}}\right] + v_2 w_2 w_{14}\left[\frac{\partial^2 f_2}{\partial x_2 \partial x_{14}}\right] + v_3 w_1 w_{14}\left[\frac{\partial^2 f_3}{\partial x_1 \partial x_{14}}\right] + v_3 w_3 w_{13}\left[\frac{\partial^2 f_3}{\partial x_3 \partial x_{13}}\right] +$$

$$v_3 w_3 w_{15}\left[\frac{\partial^2 f_3}{\partial x_{15} \partial x_3}\right] + v_4 w_1 w_{15}\left[\frac{\partial^2 f_3}{\partial x_{15} \partial x_1}\right] + v_4 w_4 w_{13}\left[\frac{\partial^2 f_4}{\partial x_4 \partial x_{13}}\right] + v_4 w_4 w_{14}\left[\frac{\partial^2 f_4}{\partial x_{14} \partial x_4}\right] +$$

$$v_5 w_4 w_{13}\left[\frac{\partial^2 f_5}{\partial x_4 \partial x_{13}}\right] + v_6 w_2 w_{14}\left[\frac{\partial^2 f_6}{\partial x_3 \partial x_{14}}\right] + v_6 w_3 w_{13}\left[\frac{\partial^2 f_6}{\partial x_3 \partial x_{13}}\right] + v_7 w_3 w_{15}\left[\frac{\partial^2 f_7}{\partial x_3 \partial x_{15}}\right] +$$

$$v_7 w_4 w_{14}\left[\frac{\partial^2 f_7}{\partial x_4 \partial x_{14}}\right] + v_{13} w_2 w_{12}\left[\frac{\partial^2 f_{13}}{\partial x_2 \partial x_{12}}\right] + v_{14} w_3 w_{12}\left[\frac{\partial^2 f_{14}}{\partial x_3 \partial x_{12}}\right] + v_{15} w_4 w_{12}\left[\frac{\partial^2 f_{15}}{\partial x_4 \partial x_{12}}\right] > 0$$

The non-zero partial derivatives of **f** associated with b is given as

$$b = \frac{\partial^2 f_2}{\partial x_{13} \partial \vartheta_m} = \delta_m x_1 > 0$$

It is observed that, for $a > 0\ and b > 0$, the results satisfy theorem 1 stated above .Thus, it is locally asymptotically stable and there exists a positive unstable equilibrium.

3.2 Numerical bifurcation

This is a qualitative change in behaviour of a dynamical system produced by varying a parameter in the equation. The state variables and the parameter descriptions are all in Tables 1 and 2 respectively. Backward bifurcation is an important phenomenon in compartmental epidemiological models. The existence of such a bifurcation suggests that the basic reproduction number itself is not sufficient enough to characterize or decide whether Malaria, Zika virus and Elephantiasis will prevail or not and if the disease will become endemic, it also depends on the initializes of the population involved. Thus, it is important to identify the backward bifurcation and establish its threshold. We carried out bifurcation analysis to study the behaviour of the model system (1) based on the results in the endemic equilibrium of the model state variables through numerical simulation over chosen parameter values. It is important to note that the existence of the bi-stability is not easy to simulate numerically. This is because a small interval of R_0 is required for the occurrence of backward bifurcation and a range of parameters had to be chosen. The qualitative bifurcation backward diagram describing the behaviour ofR_0 is presented in Fig. 1 where ϑ_m is taken as bifurcation parameter.The result indicates that backward bifurcation, if R_0 is below unity then the disease control depends on the initial sizes of the various sub-models system (1). However, reducing the R_0 below the saddle –node bifurcation value which is less than 1 but greater than zero, may result in disease eradication. However, this is guaranteed provided the disease free equilibrium is globally stable.

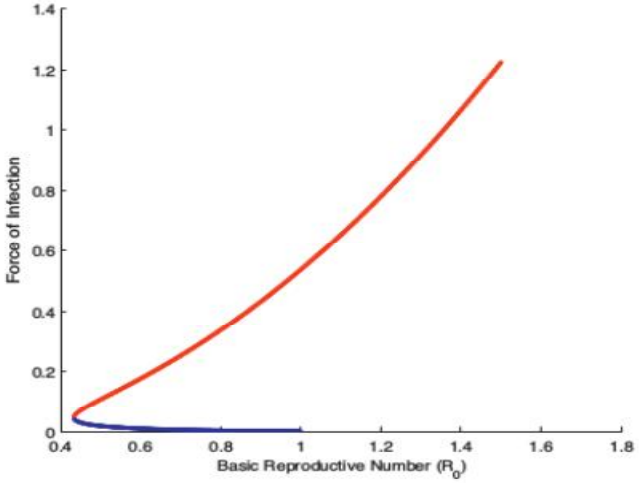

Fig. 1. Description of the backward bifurcation of the model system (1) with $\vartheta_m$as the chosen parameters

Epidemiologically, Fig. 1 implies that bringing R_0 below unity does not suffice for the eradication of multi-disease. From the analysis of the existence of the endemic equilibrium, we have established that the model system exhibits backward bifurcation when $R_0 < 1$.The existence of backward bifurcation indicates that in the neighborhood of 1, for$R_0 < 1$, a stable disease free equilibrium co-exists with two endemic equilibria, that is a smaller equilibrium (smaller number of infectious individuals) which is unstable and a larger equilibrium (with a large number of infectious individuals) which is stable. These two endemic equilibria disappear by saddle-node bifurcation when the basic reproduction number R_0 is decreased below the critical value which is less than one but greater than zero.

In order to achieve the epidemiological goal of disease eradication, R_0 must be brought below the critical value. The interpretation of this is that reducing the transmission rate or increasing treatment can lead to disappearance of the backward bifurcation curve and in this case lowering R_0 below one is sufficient to eliminate the disease from the population; A situation that will lead to forward bifurcation which is shown in Fig. 3 and lowering R_0 below unity would be sufficient to make the disease free equilibrium globally stable.

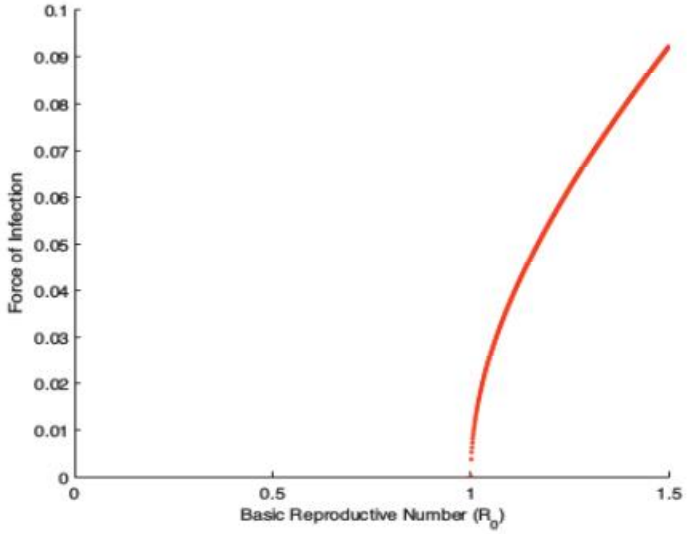

Fig. 2. Description of the forward bifurcation of the model system (1)

3.3 Sensitivity analysis

We now perform sensitivity analysis on the parameters of the model to determine which parameter will increase or decrease the basic reproduction number $\left(\mathrm{R}_0\right)$ when it is increased by a small margin. It is computed using the normalized forward sensitivity index. In terms of differentiable expression, it is defined as follows

$$S = \frac{\partial R_0}{\partial \tau_m} \times \frac{\tau_m}{R_0}.$$

Where τ_m is the parameter under consideration, Positive sensitivity index means an increase in that parameter will lead to corresponding increase in the basic reproduction number $\left(\mathrm{R}_0\right)$. However, negative sensitivity index means an increase in parameter will lead to a decrease in $\left(\mathrm{R}_0\right)$.

Table 3. Sensitivity indices for the various model only parameters

R_{ma}	Parameter values	Sensitivity index	Source
Π_h	800	-0.4655	Assumed
Π_m	1000	0.4857	[11]
μ_h	0.00004	0.4857	[12]
μ_m	0.1429	-0.9107	[13]
ϑ_m	0.034	0.9974	Assumed
δ_m	0.6502	0.9974	Assumed
τ_m	0.05	-0.4655	Assumed
η	0.05	-0.2428	Assumed
For R_{zv}			
Π_h	800	-0.4350	Assumed
ϑ_z	0.40	1.0012	Assumed
δ_z	0.12	0.10012	Assumed

R_{ma}	Parameter values	Sensitivity index	Source
τ_z	0.02	-0.4350	Assumed
Π_m	1000	0.4887	Assumed
μ_h	0.00004	0.4887	Assumed
μ_m	0.1429	0.8337	Assumed
For R_f			
Π_h	800	-0.4363	Assumed
Π_m	1000	0.4848	Assumed
μ_h	0.00004	0.4880	Assumed
μ_m	0.1429	-0.8337	Assumed
ϑ_f	0.034	0.9987	Assumed
δ_f	0.6502	0.9987	Assumed
τ_f	0.125	-0.4363	Assumed

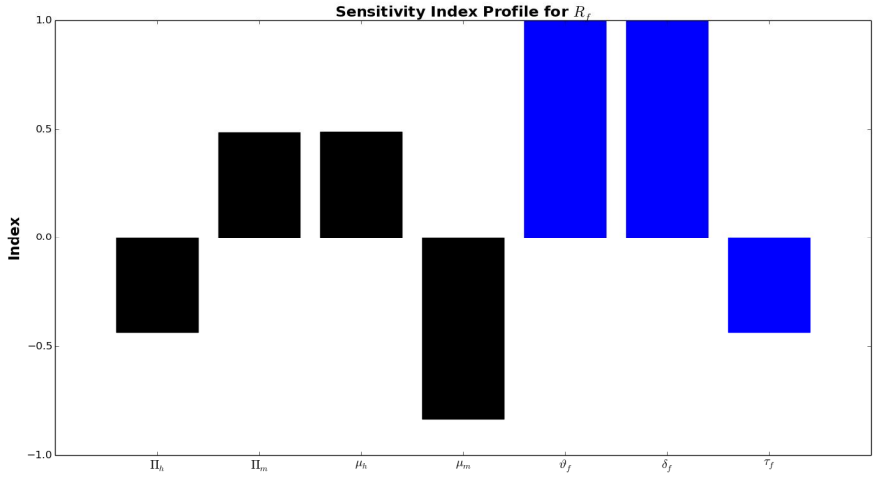

Fig. 3a.

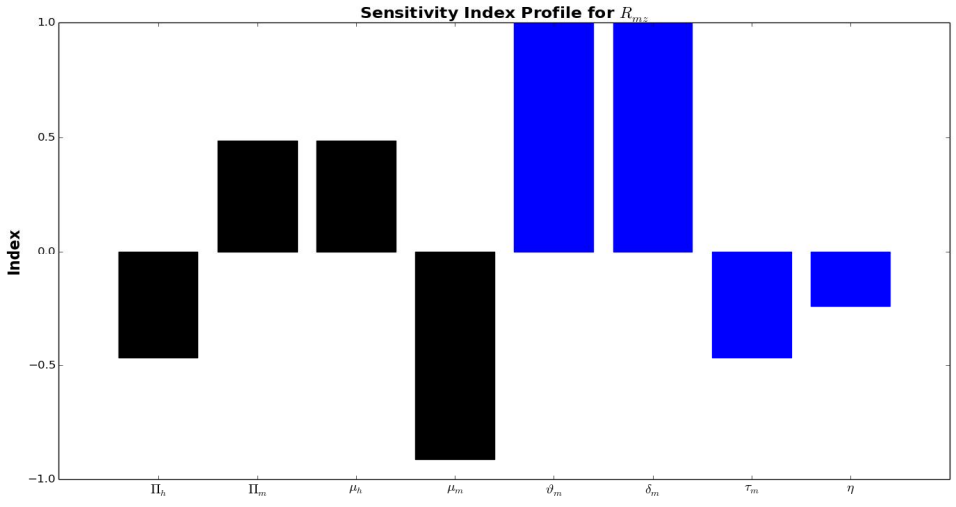

Fig. 3b.

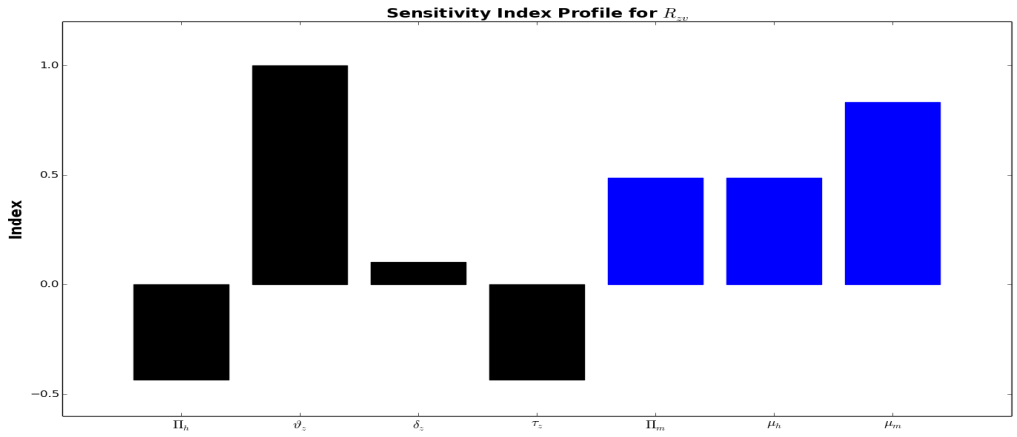

Fig. 3c.

Fig. 3: Tornado plots of partial rank correlation coefficients (PRCCs) of parameters that influence the basic reproduction number R_0 for the input using values in Table 3. Parameters with $PRCC > 0$ increases R_0 when they are increased and those with $PRCC < 0$ decreases when the R_0 is increased.

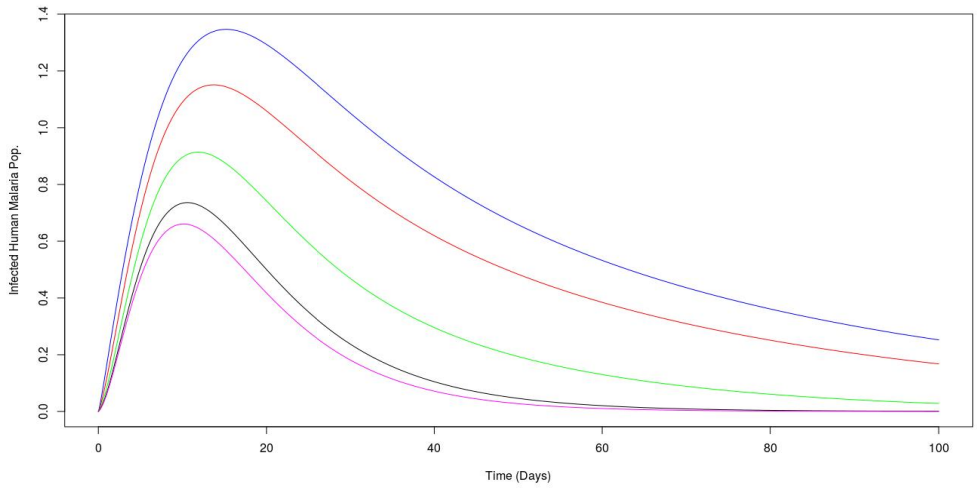

Fig. 4. Shows the changes in the number of individuals with malaria for different values of ϑ_m andδ_m

Fig. 4 track the changes in the population of individuals infected with malaria. It was observed that the infected human population with malaria will increase if the probability of transmission of infection from a mosquito with parasite infection to a susceptible human given that a contact between the two occurs. It also increases as well as if there is also an increase in the number of bites of humans per Anopheles mosquito per unit time. With parameter values shown in Table 3 and also using the following values for the state variables; $S_h = 030, I_m = 0.2, I_f = 0.2\,, I_z = 0.2, I_{mz} = 0.2, I_{mf} = 0.2, I_{zf} = 0.2, I_{mfz} = 0.2,\ R_m = 0,\ R_f = 0, R_z = 0, S_m = 0.40,\ I_p = 0.1\,, I_w = 0.1, I_a = 0.1$, we observe that when we set the values for both ϑ_m and δ_m to 0:1 the graph settles at the disease free equilibrium shown by the magenta colour and this is due to the small probability of transmission as well as the low occurrence of bites of humans by anopheles mosquito. But if the values of ϑ_m and δ_m is increased to $0.3, 0.5, 0.7, and\ 0.9$, the infected human population with malaria also increase as respectively shown by the black, green, red and blue line in the graph. Therefore, for an increases in

probability of transmission and the number of bites of humans by anopheles mosquito there will be a corresponding increase in the number of individuals infected with malaria.

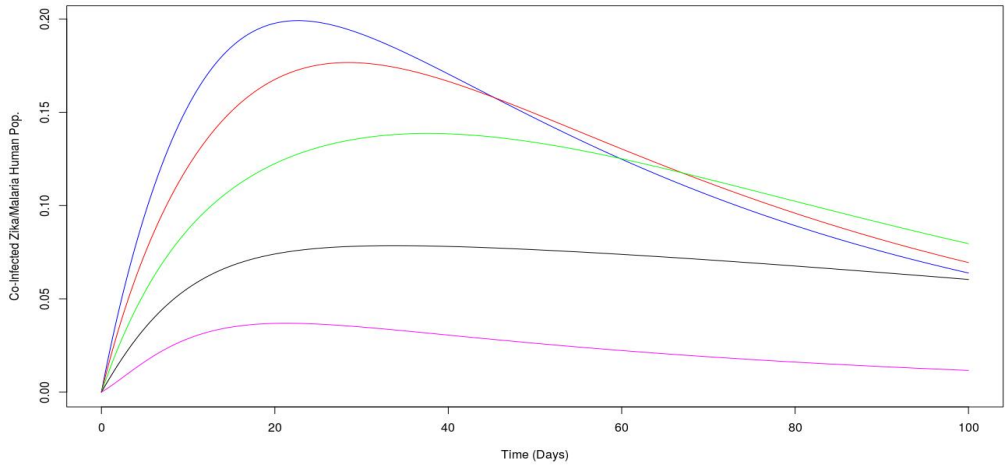

Fig. 5. Shows the changes in the number of individuals with malaria and Zika virus for different values of ϑ_m,ϑ_z, δ_m, δ_z.

For the given parameter values shown in Table 3 and using the values for the state variables shown in the explanation of Fig. 4 and also varying the values for ϑ_m,ϑ_z, δ_m and δ_z we determine their effect on the co-infected malaria and Zika compartment. It was noticed that if the probability of transmission of bothϑ_m and ϑ_z are high and relate positively with also a high number of bites of humans by mosquito (that is a corresponding high values of ϑ_m and ϑ_z) then the probability of one being infected with both disease is very high. Fig. 5 tracks this as no assertion, it was observed that when we set ϑ_m=$\vartheta_z = \delta_m = \delta_z = 0.1$ respectively the magenta line, the level of transmission is not effective and the individuals in the co-infected population tends to zero. However, the number of individuals in the co-infected population increases when the values are increased to 0.3, 0.5, 0.7 and 0.9 shown by their respective colours black, green, red and blue. In the case of the multi-infected compartment, it was observed that the value of the state variable (I_{mzf}) has more effect on the compartment compared to the major transmission parameters that has to do with the multi-infected differential equation (ϑ_m,ϑ_f, ϑ_z, δ_m, δ_f and δ_z) as shown in Fig. 6a and 6b respectively.

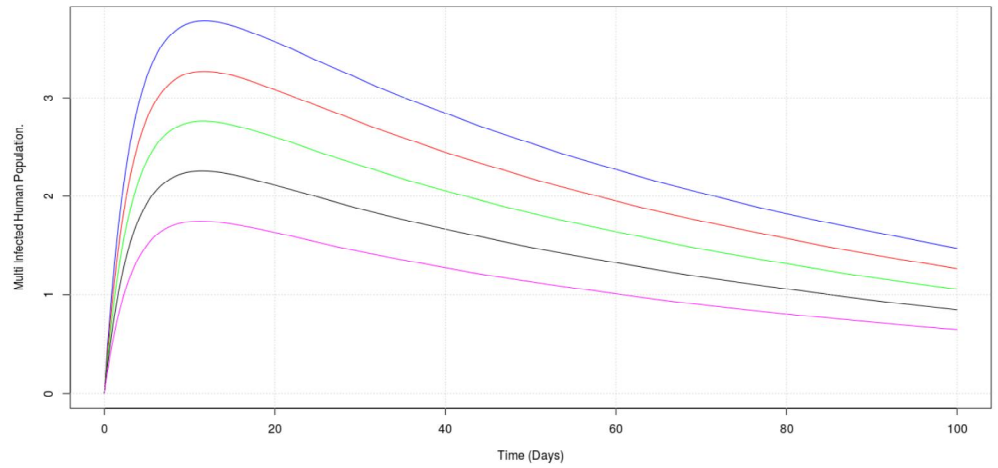

Fig. 6a. Effect of state variable values on multi-infected population

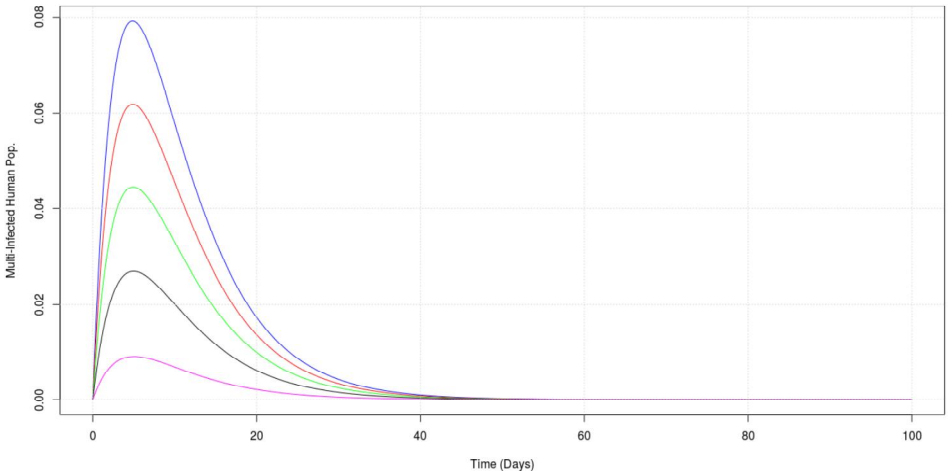

Fig. 6(b). Effect of state variable values on multi-infected population

Fig. 6 shows the effect of varying the values of the state variable (a) and parameter values (b).

From Fig. 6a, it is observed that when you set the value of the state variable of the multi-infected compartment (I_{mzf}) to 70, the disease settles at the endemic state and this increases when the value is further increased from 70 to 90,110,130 and 150 as represented by the respective magenta, black, green, red and blue line. However, in Fig. 6b, it is observed that no matter how much you increase the corresponding values for the transmission parameters for the various disease (that is malaria, elephantiasis and Zika) indicated by (ϑ_m,ϑ_f and ϑ_z)from $0.1, 0.3, 0.5, 0.7$ and 0.9 together with an increase in their corresponding number of bites per unit time (δ_m, δ_f and δ_z) also from $0.1, 0.3, 0.5, 0.7$ and 0.9 shown by the magenta, black, green, red and line respectively, the graph tends to zero.

4 Conclusion

In this article, the multi-infection model was formulated to study the transmission dynamics of Malaria, Zika virus and Elephantiasis disease in the Malaria endemic region like Kedougou in Senegal and other parts of the world that may experience multi-infection in future. Stability analysis was performed to determine both disease free and endemic equilibrium. Investigation of the existence and stability of equilibria was also performed, the model was found to exhibit backward bifurcation so that for R_0 less than unity is not sufficient to eradicate the disease from the population and there is the need to lower R_0 below a certain threshold for effective disease control. Sensitivity analysis was performed to determine parameters that have high influence on the basic reproduction number.

Competing Interests

Authors have declared that no competing interests exist

References

[1] Varandani S. Is there a cure for Zika virus? Everything you need to know about the mosquito borne disease; 2016.

[2] Peter Mpasho Mwamusaku Mwantobe. Optimal (Control of) intervention. Strategies for Malaria Epedemic in Karonga District Malawi, School of Computational and Applied; 2014.

[3] Jayner Leonard. Can you treat elephantiasis? 2018.

[4] Bhunu CP, Mushayabasa S. Transmission dynamics of lymphatic filariasis: A Mathematical Approach; 2012.

[5] The Economic Times Panache Newsletter, Zika virus: Symptoms, Treatment and Prevention.

[6] Amoah JA, Dontwi K, Bonyah E. Kwame Nkrumah University of Science and Technology, University College of Education, Kumasi Campus. Stability Analysis of Zika-Malaria Co-infection for Malaria endemic region.

[7] Hethcore HW. The mathematics of infectious diseases SIAM Rev. 2000;42(4):599-653.

[8] van den Driessche P. Watmough, reproduction numbers and sub-threshold endemic equilibra for compartmental models of disease transmission. Math, Biosci. 2000;180.

[9] Castillo-Chavez C, Blower S, van den Driessche P, Kirschner D, Abdul-Aziz Yakubu. Mathematical Approaches for Emerging and Reemerging Infectious Diseases: Models, Methods and Theory; 2002.

[10] Castillo-Chavez C, Song B. Dynamical models of tuberculosis and their applications. Mathematical Biosciences and Engineering. 2004;1:361-404.

[11] Makinde OD, Okosun KO. Impact of chemo-therapy on the optimal control of malaria disease with infected immigrants. BioSystems. 2011;104:32-41.

[12] Yang HM. Malaria transimission model for different vector of acquired immunity and temperature department parameters (vector). Revista de san'dePu'blica. 2000;34(3):223-231.

[13] Blayneh K, Cao Y, Kwon HD. Optimal control of vector – borne diseases: Treatment and prevention. Discrete and Continuous Dynamical Systems Series B. 2009;11(3):587-611.

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