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Generalized Weighted Rama Distribution: Properties and Application to Model Lifetime Data

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Authors' contributions

This work was carried out in collaboration among all authors. Author SUE designed the study, derived the properties, and wrote the first draft of the manuscript. Authors CJ and CKO were managed the analyses of the study. Author HOOI managed the literature searches and proofreading of the manuscript. All authors read and approved the final manuscript.

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Abstract

In this paper, we propose a new lifetime distribution called the generalized weighted Rama (GWR) distribution, which extends the two-parameter Rama distribution and has the Rama distribution as a special case. The GWR distribution has the ability to model data sets that have positive skewness and upside-down bathtub shape hazard rate. Expressions for mathematical and reliability properties of the GWR distribution have been derived. Estimation of parameters was achieved using the method of maximum likelihood estimation and a simulation was performed to verify the stability of the maximum likelihood estimates of the model parameters. The asymptotic confidence intervals of the parameters of the proposed distribution are obtained. The applicability of the GWR distribution was illustrated with a real data set and the results obtained show that the GWR distribution is a better candidate for the data than the other competing distributions being investigated.

Keywords: Weighted Rama distribution; hazard rate function; maximum likelihood method; order statistics.

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1 Introduction

Parametric statistical methods assume that the data being analyzed come from a particular statistical distribution. In most practical situations, some of the well-known statistical distributions do not fit **some** real life data adequately and one is led to seek some modifications of existing statistical distributions that will fit varieties of real life data more appropriately. Over the years, researchers have been developing new statistical distributions capable of providing more flexible models for several real life phenomena. One of the trending methods for generating new statistical distributions is the weighted method. The concept of weighted distributions came to limelight when Fisher [1] studied how methods of ascertainment can affect the form of distribution of recorded observations. Continuing, Rao [2] improved the weighted method in a more general manner to capture a situation where the standard distributions are inappropriate in recording statistical observations with equal probabilities, for instance, situations where the observations are non-experimental, non-replicated and non-random. Naturally, when observations are recorded according to some probabilistic mechanism, the distribution of the original observations may not have the original distribution unless every observation is given equal chance of being recorded. In view of this, the weighted method formulates models such that observations are recorded according to some weight function. A special case of the weighted distributions is the length bias distribution introduced by Cox [3] and Zelen [4], which considers the weight function as the length of the units. In particular, weighted distributions found applications in modelling biased samples in the fields of medicine [5], ecology [6], reliability and branching processes [7] and so on.

Recently, several weighted distributions have been developed and applied to many real life phenomena. Notable among them are the new class of weighted exponential distribution due to Gupta and Kundu [8], weighted Weibull proposed by Dey et al. [9], weighted exponential by Dey et al. [10], weighted Lindley introduced by Ghitany et al. [11], weighted Maxwell by Joshi and Modi [12], new weighted Lindley due to Asgharzadeh et al. [13], weighted Akash by Shanker and Shukla [14], weighted Shanker due to Shanker and Shukla [15], three-parameter weighted Lindley proposed by Shanker et al. [16], weighted exponentiated Mukherjee-Islam by Subramanian and Rather [17], two-parameter weighted Sujatha due to Shanker and Shukla [18], three-parameter weighted Pareto Type II by Para and Jan [19], two-parameter weighted Rama due to Eyob and Shanker [20], weighted quasi Akash developed by Eyob et al. [21], weighted exponential-Gompertz by Abd and Ragab [22], weighted Aradhana due to Ganaie et al. [23], weighted Sushila by Rather [24], weighted Akshaya due to Rather and Subramanian [25], Weighted Odoma by Manoj and Elangovan [26], weighted new quasi Lindley Ganaie et al. [27], weighted three-parameter Akash by Ganaie and Rajagopalan [28], weighted two parametric Rama by Vijayakumar et al. [29], weighted Suja by Alsmairan and Al-Omari [30] among others.

The aim of this paper is to introduce a generalized weighted Rama (GWR) distribution having three parameters. The paper is motivated by the fact that weighted distributions finds application in modelling biased samples, which the baseline distribution cannot offer. The remaining part of this paper is organized as follows. Section 2 gives the derivation of the probability density function and its corresponding cumulative distribution function. In Section 3, we present the stochastic ordering. Section 4 deals with the quantile function. A comprehensive account of mathematical properties of the proposed distribution is provided in Section 5. Also, reliability measures of the proposed distribution are derived in Section 6. Estimation by method of maximum likelihood and approximate confidence intervals are presented in Sections 7 and 8. In Section 9, a simulation study is presented in order to assess the performance of the estimation procedure. Section 10 illustrates an application of the proposed model by using a real data set. Finally, the conclusion of this work is provided in Section 11.

2 The Generalized Weighted Rama Distribution

From the work of Umeh et al. [31], the probability density function (pdf) and the cumulative distribution function (cdf) of the two-parameter Rama distribution having parameters θ and β may be written respectively as

$$g(x;\theta,\beta)=\frac{\theta}{\beta\theta^3+6}\left(\beta+x^3\right)e^{-\theta x};x>0,\theta>0,\beta>0 \tag{1}$$

And

$$G(x;\theta,\beta)=1-\left[1+\frac{\theta x\left(\theta^2x^2+3\theta x+6\right)}{\beta\theta^3+6}\right]e^{-\theta x};x>0,\theta>0,\beta>0 \tag{2}$$

Using the pdf (1) as the baseline distribution, we present the proposed distribution in Theorem 1.

Theorem 1. *The probability density function of the generalized weighted Rama (GWR) distribution with parameters* θ*,* β *and* λ *is given by*

$$f(x;\theta,\beta,\lambda)=\frac{\theta^{\lambda+3}}{\beta\theta^3+\lambda(\lambda+1)(\lambda+2)}\frac{x^{\lambda-1}}{\Gamma(\lambda)}\left(\beta+x^3\right)e^{-\theta x};x>0,\theta>0,\beta>0,\lambda>0 \tag{3}$$

where θ is a scale parameter, β and λ are shape parameters, and $\Gamma(\lambda)$ is the complete gamma function defined by $\Gamma(\lambda)=\int_0^\infty y^{\lambda-1}e^{-y}\,dy$.

Proof: Following the work of Eyob and Shanker [20], one defines the GWR distribution as

$$f(x;\theta,\beta,\lambda)=k\,w(x;\lambda)\,g(x;\theta,\beta) \tag{4}$$

where k is the normalizing constant, $w(x;\lambda)$ is the weighting function and $g(x;\theta,\beta)$ is the pdf of the two-parameter Rama distribution defined in (1), which serves as the baseline distribution in this work. Now, by setting $w(x;\lambda)=x^{\lambda-1}$ and putting (1) into (4), we obtain

$$f(x;\theta,\beta,\lambda)=\frac{k\,\theta^4}{\beta\theta^3+6}x^{\lambda-1}\left(\beta+x^3\right)e^{-\theta x} \tag{5}$$

Observe that (5) contains a normalizing constant k, which is to be found such that $\int_0^\infty f(x;\theta,\beta,\lambda)\,dx=1$

Thus,

$$\frac{k\,\theta^4}{\beta\theta^3+6}\int_0^\infty x^{\lambda-1}\left(\beta+x^3\right)e^{-\theta x}\,dx=1 \tag{6}$$

Solving (6) for k gives

$$k=\frac{\left(\beta\theta^3+6\right)\theta^{\lambda}}{\theta\left[\beta\theta^3\Gamma(\lambda)+\Gamma(\lambda+3)\right]} \tag{7}$$

Substituting (7) into (5), one obtains the pdf of the GWR distribution defined in (3) and this completes the proof of Theorem 1.

Observe that the pdf (3) can be expressed as a two-component mixture of gamma(θ,λ) and gamma$(\theta,\lambda+3)$ distributions such that

$$f(x;\theta,\beta,\lambda)=ph_1(x;\theta,\lambda)+(1-p)h_2(x;\theta,\lambda+3) \tag{8}$$

where

$$p=\frac{\beta\theta^3}{\beta\theta^3+\lambda(\lambda+1)(\lambda+2)},\ h_1(x;\theta,\lambda)=\frac{\theta^{\lambda}x^{\lambda-1}e^{-\theta x}}{\Gamma(\lambda)}\ \text{and}\ h_2(x;\theta,\lambda+3)=\frac{\theta^{\lambda+3}x^{(\lambda+3)-1}e^{-\theta x}}{\Gamma(\lambda+3)}$$

Corollary 1. *The area under the curve of the GWR distribution is unity.*

Proof: We are required to show that $\int_0^\infty f(x;\theta,\beta,\lambda)\,dx=1$. Thus

$$\int_0^\infty f(x;\theta,\beta,\lambda)\,dx=\frac{\theta^{\lambda+3}}{\beta\theta^3+\lambda(\lambda+1)(\lambda+2)}\frac{1}{\Gamma(\lambda)}\int_0^\infty x^{\lambda-1}\left(\beta+x^3\right)e^{-\theta x}$$

$$=\frac{\theta^{\lambda+3}}{\beta\theta^3+\lambda(\lambda+1)(\lambda+2)}\frac{1}{\Gamma(\lambda)}\left[\beta\int_0^\infty x^{\lambda-1}e^{-\theta x}\,dx+\int_0^\infty x^{(\lambda+3)-1}e^{-\theta x}\,dx\right]$$

$$=\frac{\theta^{\lambda+3}}{\beta\theta^3+\lambda(\lambda+1)(\lambda+2)}\frac{1}{\Gamma(\lambda)}\left[\frac{\beta\Gamma(\lambda)}{\theta^{\lambda}}+\frac{\Gamma(\lambda+3)}{\theta^{\lambda+3}}\right]\quad\because\int_0^\infty y^{\alpha-1}e^{-\beta y}\,dy=\frac{\Gamma(\alpha)}{\beta^{\alpha}}$$

$$=\frac{\theta^{\lambda+3}}{\beta\theta^3+\lambda(\lambda+1)(\lambda+2)}\frac{1}{\Gamma(\lambda)}\left\{\frac{\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]\Gamma(\lambda)}{\theta^{\lambda+3}}\right\}=1$$

Theorem 2. *The cumulative distribution function of the GWR distribution with parameters θ, β and λ is given by*

$$F(x;\theta,\beta,\lambda)=1-\left[\frac{\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]\Gamma(\lambda,\theta x)+\left[(\theta x)^2+(\lambda+2)(\theta x)+(\lambda+1)(\lambda+2)\right](\theta x)^{\lambda}e^{-\theta x}}{\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]\Gamma(\lambda)}\right] \quad (9)$$

for $x>0,\theta>0,\beta>0,\lambda>0$ and $\Gamma(\lambda,\theta x)=\int_{\theta x}^{\infty} y^{\lambda-1}e^{-y}\,dy$ is an upper incomplete gamma function.

Proof: The cumulative distribution function (cdf) of the GWR distribution (3) is obtained as follows

$$F(x;\theta,\beta,\lambda)=\frac{\theta^{\lambda+3}}{\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]\Gamma(\lambda)}\int_0^x x^{\lambda-1}\left(\beta+x^3\right)e^{-\theta x}\,dx$$

Letting $y=\theta x$, $x=y/\theta$ and $dx=dy/\theta$, then $x\in(0,x)\Rightarrow y\in(0,\theta x)$ and thus,

$$F(x;\theta,\beta,\lambda)=\frac{\theta^{\lambda+3}}{\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]\Gamma(\lambda)}\left[\frac{\beta}{\theta^{\lambda}}\int_0^{\theta x} y^{\lambda-1}e^{-y}\,dy+\frac{1}{\theta^{\lambda+3}}\int_0^{\theta x} y^{(\lambda+3)-1}e^{-y}\,dy\right]$$

$$=\frac{\beta\theta^3\gamma(\lambda,\theta x)+\gamma(\lambda+3,\theta x)}{\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]\Gamma(\lambda)} \quad (10)$$

where $\gamma(\lambda,\theta x)=\int_0^{\theta x} y^{\lambda-1}e^{-y}\,dy$ and $\gamma(\lambda+3,\theta x)=\int_0^{\theta x} y^{(\lambda+3)-1}e^{-y}\,dy$ are lower incomplete gamma functions. Equation (10) may be simplified by recalling that

$$\Gamma(s)=\gamma(s,x)+\Gamma(s,x)$$

$$\Rightarrow \gamma(\lambda,\theta x)=\Gamma(\lambda)-\Gamma(\lambda,\theta x) \quad (11)$$

$$\Rightarrow \gamma(\lambda+3,\theta x)=\Gamma(\lambda+3)-\Gamma(\lambda+3,\theta x) \quad (12)$$

To obtain the expression for $\Gamma(\lambda+3,\theta x)$ of (12), we recall that

$$\Gamma(\lambda+1,x)=\lambda\Gamma(\lambda,x)+x^{\lambda}e^{-x}$$

$$\Rightarrow \Gamma(\lambda+1,\theta x)=\lambda\Gamma(\lambda,\theta x)+(\theta x)^{\lambda}e^{-\theta x} \quad (13)$$

$$\Rightarrow \Gamma(\lambda+2,\theta x)=\lambda(\lambda+1)\Gamma(\lambda,\theta x)+(\lambda+1)(\theta x)^{\lambda}e^{-\theta x}+(\theta x)^{\lambda+1}e^{-\theta x} \quad (14)$$

$$\Rightarrow \Gamma(\lambda+3,\theta x)=\lambda(\lambda+1)(\lambda+2)\Gamma(\lambda,\theta x)+\left[(\lambda+1)(\lambda+2)+(\lambda+2)(\theta x)+(\theta x)^2\right](\theta x)^{\lambda}e^{-\theta x} \quad (15)$$

Now, a substitution of (15) into (12) leads to

$$\gamma(\lambda+3,\theta x)=\lambda(\lambda+1)(\lambda+2)\Gamma(\lambda)-\left\{\begin{array}{l}\lambda(\lambda+1)(\lambda+2)\Gamma(\lambda,\theta x)\\+\left[(\lambda+1)(\lambda+2)+(\lambda+2)(\theta x)+(\theta x)^2\right](\theta x)^{\lambda}e^{-\theta x}\end{array}\right\} \tag{16}$$

Putting (11) and (16) into (10), we obtain

$$F(x)=\frac{\begin{array}{l}\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]\Gamma(\lambda)+\lambda(\lambda+1)(\lambda+2)\Gamma(\lambda)\\-\left\{\lambda(\lambda+1)(\lambda+2)\Gamma(\lambda,\theta x)+\left[(\lambda+1)(\lambda+2)+(\lambda+2)(\theta x)+(\theta x)^2\right](\theta x)^{\lambda}e^{-\theta x}\right\}\end{array}}{\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]\Gamma(\lambda)} \tag{17}$$

Further simplification of (17) leads to (9), which completes the proof of Theorem 2.

The various shapes of the pdf and cdf of the GWR distribution are shown in Figs. 1 and 2 respectively.

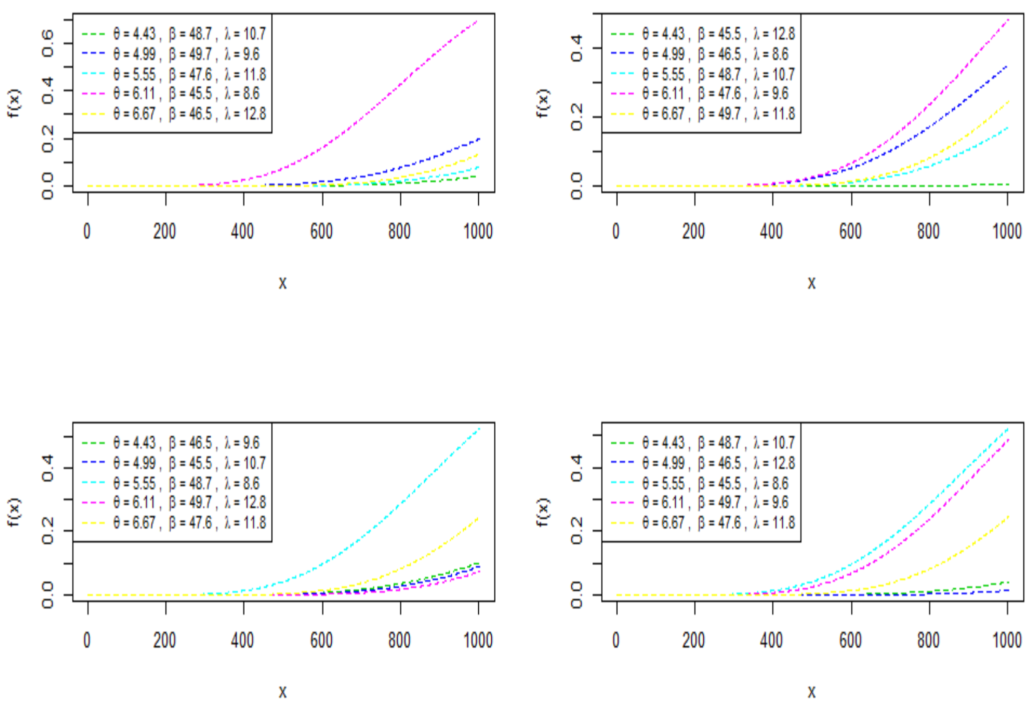

Fig. 1. Density function shapes for GWR distribution considering different values of θ, β and λ

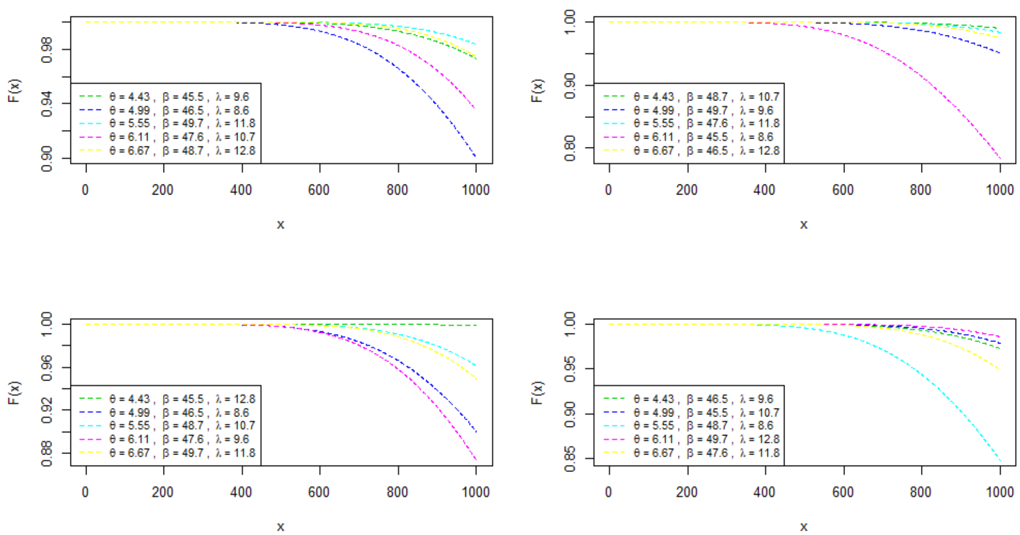

Fig. 2. Cumulative distribution function shapes for GWR distribution considering different values of θ, β and λ

3 Stochastic Ordering of The GWR Distribution

One of the activities of distribution theorists deals with the comparison of random quantities based on some measures associated to them. A good index for achieving such comparison is the stochastic ordering, which is routinely used in many applications in economics, finance, insurance, management science, operations research, statistics, and other various fields of study (Shaked and Shanthikumar [32]). According to Dey et al. [33], the different types of stochastic orderings which are useful in ordering random variables include the usual stochastic order, the hazard rate order, the mean residual life order and the likelihood ratio order. Given two nonnegative random variables X and Y with absolutely continuous distribution functions $F_X(x)$ and $F_Y(x)$, hazard rate functions $h_X(x)$ and $h_Y(x)$, and mean residual life functions $m_X(x)$ and $m_Y(x)$ respectively, then X is said to be smaller than Y in the(i) usual stochastic order $(X \leq_{st} Y)$ if $F_X(x) \geq F_Y(x)$ for all x; (ii) hazard rate order $(X \leq_{hr} Y)$ if $h_X(x) \geq h_Y(x)$ for all x; (iii) mean residual life order $(X \leq_{mrl} Y)$ if $m_X(x) \geq m_Y(x)$ for all x; and

(iv) likelihood ratio order $(X \leq_{lr} Y)$ if $f_X(x)/f_Y(x)$ decreases in x.

It may be noted that a distribution that has likelihood ratio (lr) ordering possesses the same ordering in hazard rate (hr) and distribution (st). In addition, if a family of distributions has likelihood ratio ordering, then there exists a uniformly most powerful test (Shaked and Shanthikumar [32]). Meanwhile, the following stochastic ordering relationships due to Shaked and Shanthikumar [34] are well known for establishing stochastic ordering of continuous distributions

$$X \leq_{lr} Y \Rightarrow X \underset{\underset{X \leq_{st} Y}{\Downarrow}}{\leq_{hr}} Y \Rightarrow X \leq_{mlr} Y$$

In view of the above relationship, it is clear that the likelihood ratio appears stronger than the other ones and can be used as a sufficient condition for the rest of the other stochastic orders (Belzunce [35]). Consequently, the GWR distribution is said to be ordered with respect to the strongest "likelihood ratio" ordering as shown in Theorem 3.

Theorem 3. *Let X and Y follow a generalized weighted Rama distribution with parameters $(\theta_1, \lambda_1, \beta_1)$ and $(\theta_2, \lambda_2, \beta_2)$, then*

(i) $\lambda_1 = \lambda_2, \beta_1 = \beta_2$ and $\theta_1 > \theta_2$, then $X \leq_{lr} Y$, $X \leq_{hr} Y$, $X \leq_{mlr} Y$ and $X \leq_{st} Y$

(ii) $\lambda_1 < \lambda_2, \beta_1 = \beta_2$ and $\theta_1 = \theta_2$, then $X \leq_{lr} Y$, $X \leq_{hr} Y$, $X \leq_{mlr} Y$ and $X \leq_{st} Y$

(iii) $\lambda_1 = \lambda_2, \beta_1 > \beta_2$ and $\theta_1 = \theta_2$, then $X \leq_{lr} Y$, $X \leq_{hr} Y$, $X \leq_{mlr} Y$ and $X \leq_{st} Y$

(iv) $\lambda_1 < \lambda_2, \beta_1 > \beta_2$ and $\theta_1 > \theta_2$, then $X \leq_{lr} Y$, $X \leq_{hr} Y$, $X \leq_{mlr} Y$ and $X \leq_{st} Y$

Proof: The likelihood ratio for the GWR distribution is

$$\frac{f_X(x;\theta_1,\lambda_1,\beta_1)}{f_Y(x;\theta_2,\lambda_2,\beta_2)} = \frac{\theta_1^{\lambda_1+3}}{\theta_2^{\lambda_2+3}} \frac{\beta_2\theta_2^3 + \lambda_2(\lambda_2+1)(\lambda_2+2)}{\beta_1\theta_1^3 + \lambda_1(\lambda_1+1)(\lambda_1+2)} \frac{\Gamma(\lambda_2)}{\Gamma(\lambda_1)} \left(\frac{\beta_1 + x^3}{\beta_2 + x^3}\right) x^{\lambda_1-\lambda_2} e^{-(\theta_1-\theta_2)x} \tag{18}$$

Next, the log-likelihood ratio may be expressed as

$$\begin{aligned}\ln\left(\frac{f_X(x;\theta_1,\lambda_1,\beta_1)}{f_Y(x;\theta_2,\lambda_2,\beta_2)}\right) &= \ln\left(\frac{\theta_1^{\lambda_1+3}}{\theta_2^{\lambda_2+3}} \frac{\beta_2\theta_2^3 + \lambda_2(\lambda_2+1)(\lambda_2+2)}{\beta_1\theta_1^3 + \lambda_1(\lambda_1+1)(\lambda_1+2)} \frac{\Gamma(\lambda_2)}{\Gamma(\lambda_1)}\right) + (\lambda_1 - \lambda_2)\ln x \\ &\quad + \ln(\beta_1 + x^3) - \ln(\beta_2 + x^3) - (\theta_1 - \theta_2)x\end{aligned} \tag{19}$$

Differentiating (19) with respect to x, we obtain

$$\frac{d}{dx}\ln\left(\frac{f_X(x;\theta_1,\lambda_1,\beta_1)}{f_Y(x;\theta_2,\lambda_2,\beta_2)}\right) = \frac{\lambda_1 - \lambda_2}{x} + \frac{3(\beta_2 - \beta_1)x^2}{(\beta_1 + x^3)(\beta_2 + x^3)} - (\theta_1 - \theta_2) \tag{20}$$

For $\lambda_1 = \lambda_2, \beta_1 = \beta_2$ and $\theta_1 > \theta_2$, one obtains

$$\frac{d}{dx}\ln\left(\frac{f_X(x;\theta_1,\lambda_1,\beta_1)}{f_Y(x;\theta_2,\lambda_2,\beta_2)}\right) = -(\theta_1 - \theta_2) < 0$$

Also, for $\lambda_1 < \lambda_2, \beta_1 = \beta_2$ and $\theta_1 = \theta_2$, we get

$$\frac{d}{dx}\ln\left(\frac{f_X\left(x;\theta_1,\lambda_1,\beta_1\right)}{f_Y\left(x;\theta_2,\lambda_2,\beta_2\right)}\right)=\frac{\lambda_1-\lambda_2}{x}<0$$

Again, if $\lambda_1=\lambda_2, \beta_1>\beta_2$ and $\theta_1=\theta_2$, one obtains

$$\frac{d}{dx}\ln\left(\frac{f_X\left(x;\theta_1,\lambda_1,\beta_1\right)}{f_Y\left(x;\theta_2,\lambda_2,\beta_2\right)}\right)=\frac{3\left(\beta_2-\beta_1\right)x^2}{\left(\beta_1+x^3\right)\left(\beta_2+x^3\right)}<0$$

Finally, when $\lambda_1<\lambda_2, \beta_1>\beta_2$ and $\theta_1>\theta_2$, the following result is obtained

$$\frac{d}{dx}\ln\left(\frac{f_X\left(x;\theta_1,\lambda_1,\beta_1\right)}{f_Y\left(x;\theta_2,\lambda_2,\beta_2\right)}\right)=\frac{\lambda_1-\lambda_2}{x}+\frac{3\left(\beta_2-\beta_1\right)x^2}{\left(\beta_1+x^3\right)\left(\beta_2+x^3\right)}-\left(\theta_1-\theta_2\right)<0$$

Since $\frac{d}{dx}\ln\left(\frac{f_X\left(x;\theta_1,\lambda_1,\beta_1\right)}{f_Y\left(x;\theta_2,\lambda_2,\beta_2\right)}\right)<0$ for conditions (i), (ii), (iii) and (iv), then $X\leq_{lr}Y$ and hence $X\leq_{hr}Y$, $X\leq_{mlr}Y$ and $X\leq_{st}Y$, which completes the proof of Theorem 3.

4 Quantile Function

Let X be a generalized Weighted Rama distributed random variable having cdf (9), then, the pth quantile function of X, denoted by $x_p=Q(p)=F^{-1}(p)$, can be obtained by inverting (9), as follows

$$x_p=-\frac{1}{\theta}\ln\left\{\frac{\left[(1+p)\Gamma(\lambda)-\Gamma(\lambda,\theta x)\right]\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]}{\left[\left(\theta x_p\right)^2+(\lambda+2)\left(\theta x_p\right)+(\lambda+1)(\lambda+2)\right]\left(\theta x_p\right)^\lambda}\right\} \tag{21}$$

where $p\in(0,1)$. The quantile function (21) is useful for generating random numbers from the GWR distribution.

5 Mathematical Properties of the Generalized Weighted Rama Distribution

In this section, important properties of the GWR distribution including the raw moments, central moments, coefficient of variation, index of dispersions, skewness, kurtosis, harmonic mean, Bonferroni curve, Lorenz curve, entropy, distribution of order statistics, moment generating function and characteristic function are presented.

5.1 Raw Moments of the Generalized Weighted Rama Distribution

Theorem 4. *For the random variable X having the GWR distribution with parameters θ, β and λ, the rth raw moment is given by*

$$\mu_r' = \frac{\Gamma(\lambda+r)}{\Gamma(\lambda)}\left[\frac{\beta\theta^3+(\lambda+r)(\lambda+r+1)(\lambda+r+2)}{\theta^r\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]}\right] \tag{22}$$

Proof: The rth raw moment of a continuous random variable X is given by

$$\mu_r' = E\left(X^r\right) = \int_{-\infty}^{\infty} x^r f(x)\,dx \tag{23}$$

Substituting (3) into (23), the rth raw moment of the GWR distribution is obtained as follows

$$\mu_r' = \frac{\theta^{\lambda+3}}{\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]\Gamma(\lambda)}\int_0^{\infty} x^{\lambda-1}\left(\beta+x^3\right)e^{-\theta x}\,dx$$

$$= \frac{\theta^{\lambda+3}}{\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]\Gamma(\lambda)}\left[\beta\int_0^{\infty} x^{\lambda-1}e^{-\theta x}\,dx+\int_0^{\infty} x^{(\lambda+3)-1}e^{-\theta x}\,dx\right]$$

$$= \frac{\theta^{\lambda+3}}{\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]\Gamma(\lambda)}\left[\frac{\beta\Gamma(\lambda+r)}{\theta^{\lambda+r}}+\frac{\beta\Gamma(\lambda+r+3)}{\theta^{\lambda+r+3}}\right] \tag{24}$$

After little algebra on (24), one gets the results in (22) and the proof is complete.

In particular, the first four raw moments of GWR distribution are respectively defined as:

$$\mu_1' = \frac{\lambda\left[\beta\theta^3+(\lambda+1)(\lambda+2)(\lambda+3)\right]}{\theta\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]} \tag{25}$$

$$\mu_2' = \frac{\lambda(\lambda+1)\left[\beta\theta^3+(\lambda+2)(\lambda+3)(\lambda+4)\right]}{\theta^2\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]} \tag{26}$$

$$\mu_3' = \frac{\lambda(\lambda+1)(\lambda+2)\left[\beta\theta^3+(\lambda+3)(\lambda+4)(\lambda+5)\right]}{\theta^3\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]} \tag{27}$$

$$\mu_4' = \frac{\lambda(\lambda+1)(\lambda+2)(\lambda+3)\left[\beta\theta^3+(\lambda+4)(\lambda+5)(\lambda+6)\right]}{\theta^4\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]} \tag{28}$$

5.2 Central Moments of the Generalized Weighted Rama Distribution

Theorem 5. *For the random variable X having the GWR distribution with parameters θ, β and λ, the rth central moment is given by*

$$\mu_r = \sum_{i=0}^{r} \binom{r}{i} \frac{(-1)^{r-i}\left(\lambda\left[\beta\theta^3+(\lambda+1)(\lambda+2)(\lambda+3)\right]\right)^{r-i}\left[\beta\theta^3+(\lambda+i)(\lambda+i+1)(\lambda+i+2)\right]\Gamma(\lambda+i)}{\theta^r\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]^{r-i+1}} \tag{28}$$

Proof: The rth central moment of a GWR variable X is given by

$$\mu_r = E\left[(x-\mu)^r\right] = \int_{-\infty}^{\infty}(x-\mu)^r f(x;\theta,\beta,\lambda)\,dx = \sum_{i=0}^{r}\binom{r}{i}(-\mu)^{r-i}E\left(X^i\right)$$

$$= \sum_{i=0}^{r}\binom{r}{i}\left(-\frac{\lambda\left[\beta\theta^3+(\lambda+1)(\lambda+2)(\lambda+3)\right]}{\theta\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]}\right)^{r-i} \times \frac{\Gamma(\lambda+i)}{\Gamma(\lambda)}\left[\frac{\beta\theta^3+(\lambda+i)(\lambda+i+1)(\lambda+i+2)}{\theta^i\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]}\right] \tag{29}$$

Further simplification of (29) yields (28), which completes the proof of Theorem 5.

It is obvious that the first central moment is zero while the second, third and fourth central moments of the GWR distribution are respectively given by

$$\mu_2 = \sum_{i=0}^{2} \binom{2}{i} \frac{(-1)^{2-i}\left(\lambda\left[\beta\theta^3+(\lambda+1)(\lambda+2)(\lambda+3)\right]\right)^{2-i}\left[\beta\theta^3+(\lambda+i)(\lambda+i+1)(\lambda+i+2)\right]\Gamma(\lambda+i)}{\theta^2\Gamma(\lambda)\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]^{3-i}} \tag{30}$$

$$\mu_3 = \sum_{i=0}^{3} \binom{3}{i} \frac{(-1)^{3-i}\left(\lambda\left[\beta\theta^3+(\lambda+1)(\lambda+2)(\lambda+3)\right]\right)^{3-i}\left[\beta\theta^3+(\lambda+i)(\lambda+i+1)(\lambda+i+2)\right]\Gamma(\lambda+i)}{\theta^3\Gamma(\lambda)\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]^{4-i}} \tag{31}$$

$$\mu_4 = \sum_{i=0}^{4} \binom{4}{i} \frac{(-1)^{4-i}\left(\lambda\left[\beta\theta^3+(\lambda+1)(\lambda+2)(\lambda+3)\right]\right)^{4-i}\left[\beta\theta^3+(\lambda+i)(\lambda+i+1)(\lambda+i+2)\right]\Gamma(\lambda+i)}{\theta^4\Gamma(\lambda)\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]^{5-i}} \tag{32}$$

Observe that the first raw moment is the mean $E(X) = \mu_1^{'}$ and the second central moment is the variance $Var(X) = \sigma^2$, which may be alternatively expressed as

$$Var(X) = \mu_2^1 - \left(\mu_1^1\right)^2 = \frac{\lambda(\lambda+1)\left[\beta\theta^3+(\lambda+2)(\lambda+3)(\lambda+4)\right]}{\theta^2\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]} - \left(\frac{\lambda\left[\beta\theta^3+(\lambda+1)(\lambda+2)(\lambda+3)\right]}{\theta\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]}\right)^2 \tag{33}$$

5.3 Coefficient of Variation and Index of Dispersion of the Generalized Weighted Rama Distribution

Given the random variable X having the GWR distribution with parameters θ, β and λ, then the coefficient of variation (cv) is given by

$$cv=\frac{\sigma}{\mu}=\frac{\left[\frac{\lambda(\lambda+1)\left[\beta\theta^3+(\lambda+2)(\lambda+3)(\lambda+4)\right]}{\theta^2\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]}-\left(\frac{\lambda\left[\beta\theta^3+(\lambda+1)(\lambda+2)(\lambda+3)\right]}{\theta\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]}\right)^2\right]^{\frac{1}{2}}}{\frac{\lambda\left[\beta\theta^3+(\lambda+1)(\lambda+2)(\lambda+3)\right]}{\theta\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]}} \tag{34}$$

Similarly, the index of dispersion (γ) of the GWR distributed random variable X is given by

$$\gamma=\frac{\sigma^2}{\mu}=\frac{\frac{\lambda(\lambda+1)\left[\beta\theta^3+(\lambda+2)(\lambda+3)(\lambda+4)\right]}{\theta^2\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]}-\left(\frac{\lambda\left[\beta\theta^3+(\lambda+1)(\lambda+2)(\lambda+3)\right]}{\theta\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]}\right)^2}{\frac{\lambda\left[\beta\theta^3+(\lambda+1)(\lambda+2)(\lambda+3)\right]}{\theta\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]}} \tag{35}$$

5.4 Skewness and Kurtosis of the Generalized Weighted Rama Distribution

Given the random variable X having the GWR distribution with parameters θ, β and λ, then the skewness and kurtosis are respectively given by

$$\sqrt{\beta_1}=\frac{\mu_3}{\sigma^3}=\frac{\sum_{i=0}^{3}\binom{3}{i}\frac{(-1)^{3-i}\left(\lambda\left[\beta\theta^3+(\lambda+1)(\lambda+2)(\lambda+3)\right]\right)^{3-i}\left[\beta\theta^3+(\lambda+i)(\lambda+i+1)(\lambda+i+2)\right]\Gamma(\lambda+i)}{\theta^3\Gamma(\lambda)\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]^{4-i}}}{\left[\frac{\lambda(\lambda+1)\left[\beta\theta^3+(\lambda+2)(\lambda+3)(\lambda+4)\right]}{\theta^2\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]}-\left(\frac{\lambda\left[\beta\theta^3+(\lambda+1)(\lambda+2)(\lambda+3)\right]}{\theta\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]}\right)^2\right]^{\frac{3}{2}}} \tag{36}$$

and

$$\beta_2=\frac{\mu_4}{\sigma^4}=\frac{\sum_{i=0}^{4}\binom{4}{i}\frac{(-1)^{4-i}\left(\lambda\left[\beta\theta^3+(\lambda+1)(\lambda+2)(\lambda+3)\right]\right)^{4-i}\left[\beta\theta^3+(\lambda+i)(\lambda+i+1)(\lambda+i+2)\right]\Gamma(\lambda+i)}{\theta^4\Gamma(\lambda)\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]^{5-i}}}{\left[\frac{\lambda(\lambda+1)\left[\beta\theta^3+(\lambda+2)(\lambda+3)(\lambda+4)\right]}{\theta^2\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]}-\left(\frac{\lambda\left[\beta\theta^3+(\lambda+1)(\lambda+2)(\lambda+3)\right]}{\theta\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]}\right)^2\right]^{2}} \tag{37}$$

The mean, variance, skewness, kurtosis, coefficient of variation and index of dispersion of the GWRD for different values of θ, β and λ are given in Tables 1-5. It can be seen that the mean, variance and index of diversity decreases as the value of θ and β increases for fixed λ. The positive values of skewness mean that the GWRD is skewed to the right.

Table 1. Mean, variance, Skewness, Kurtosis, coefficient of variation and index of dispersion for the GWR distribution

λ	θ	β	Mean	Variance	Skewness	Kurtosis	CV	γ
1	4.5	0.5	0.2998	0.1123	2.2169	9.6195	1.1179	0.3747
1	4.5	1.5	0.2503	0.0735	2.3872	11.6617	1.0834	0.2938
1	4.5	3.5	0.2345	0.0602	2.2835	11.259	1.0459	0.2566
1	4.5	5.5	0.2301	0.0563	2.2119	10.7619	1.0314	0.2448
1	4.5	10.0	0.2266	0.0532	2.1331	10.146	1.0183	0.2350
1	5.0	0.5	0.2526	0.0793	2.3162	10.5186	1.1149	0.3139
1	5.0	1.5	0.2186	0.0545	2.3606	11.6571	1.0683	0.2495
1	5.0	3.5	0.2081	0.0464	2.2328	10.9146	1.0353	0.2231
1	5.0	5.5	0.2052	0.0441	2.1672	10.4197	1.0237	0.2150
1	5.0	10.0	0.2029	0.0423	2.1016	9.8846	1.0136	0.2084
1	7.5	0.5	0.1444	0.0236	2.3459	11.6024	1.0629	0.1631
1	7.5	1.5	0.1371	0.0198	2.1783	10.5065	1.0256	0.1442
1	7.5	3.5	0.135	0.0186	2.0876	9.7670	1.0116	0.1381
1	7.5	5.5	0.1344	0.0183	2.0579	9.5123	1.0075	0.1364
1	7.5	10.0	0.1339	0.0181	2.0329	9.2929	1.0042	0.1350
1	10.0	0.5	0.1036	0.0114	2.2123	10.7645	1.0315	0.1102
1	10.0	1.5	0.1012	0.0105	2.0864	9.7566	1.0115	0.1035
1	10.0	3.5	0.1005	0.0102	2.0393	9.3496	1.0050	0.1015
1	10.0	5.5	0.1003	0.0101	2.0254	9.2271	1.0032	0.1010
1	10.0	10.0	0.1002	0.0101	2.0142	9.1270	1.0018	0.1005

Table 2. Mean, variance, Skewness, Kurtosis, coefficient of variation and Index of dispersion for the GWR distribution

λ	θ	β	Mean	Variance	Skewness	Kurtosis	CV	γ
2	4.5	0.5	0.6745	0.2503	1.2323	4.7676	0.7418	0.3711
2	4.5	1.5	0.5440	0.1774	1.5768	6.4225	0.7741	0.3260
2	4.5	3.5	0.4911	0.1381	1.6448	7.0921	0.7566	0.2811
2	4.5	5.5	0.4749	0.1249	1.6210	7.0894	0.7442	0.2630
2	4.5	10.0	0.4616	0.1137	1.5660	6.8687	0.7305	0.2463
2	5.0	0.5	0.5665	0.1855	1.3508	5.2328	0.7602	0.3274
2	5.0	1.5	0.4681	0.1298	1.6240	6.7807	0.7698	0.2774
2	5.0	3.5	0.4312	0.1040	1.6308	7.1111	0.7478	0.2412
2	5.0	5.5	0.4202	0.0958	1.5931	6.9907	0.7365	0.2279
2	5.0	10.0	0.4113	0.0889	1.5366	6.7201	0.7250	0.2162
2	7.5	0.5	0.3075	0.0557	1.6346	6.8824	0.7673	0.1811
2	7.5	1.5	0.2813	0.0431	1.6009	7.0217	0.7384	0.1534
2	7.5	3.5	0.2731	0.0389	1.5223	6.6435	0.7225	0.1426
2	7.5	5.5	0.2708	0.0377	1.4892	6.4572	0.7173	0.1393
2	7.5	10.0	0.2689	0.0368	1.4584	6.2749	0.7129	0.1367
2	10.0	0.5	0.2137	0.0253	1.6212	7.0899	0.7443	0.1184
2	10.0	1.5	0.2047	0.0219	1.5210	6.6364	0.7223	0.1068
2	10.0	3.5	0.2020	0.0208	1.4666	6.3239	0.7140	0.1030
2	10.0	5.5	0.2013	0.0205	1.4488	6.2163	0.7116	0.1019
2	10.0	10.0	0.2007	0.0203	1.4338	6.1236	0.7096	0.1011

Table 3. Mean, variance, Skewness, Kurtosis, coefficient of variation and Index of dispersion for the GWR distribution

λ	θ	β	Mean	Variance	Skewness	Kurtosis	CV	γ
3	4.5	0.5	1.0456	0.3414	0.8323	3.7890	0.5588	0.3265
3	4.5	1.5	0.8700	0.2876	1.1234	4.5689	0.6164	0.3305
3	4.5	3.5	0.7722	0.2308	1.2870	5.3322	0.6222	0.2989
3	4.5	5.5	0.7379	0.2064	1.3159	5.5663	0.6157	0.2797
3	4.5	10.0	0.7079	0.1831	1.3040	5.6301	0.6044	0.2586
3	5.0	0.5	0.8939	0.2687	0.9087	3.9366	0.5799	0.3006
3	5.0	1.5	0.7455	0.2152	1.1994	4.8778	0.6223	0.2887
3	5.0	3.5	0.6724	0.1727	1.3114	5.5137	0.6180	0.2568
3	5.0	5.5	0.6482	0.1562	1.3150	5.6310	0.6098	0.2410
3	5.0	10.0	0.6275	0.1412	1.2852	5.5825	0.5989	0.2251
3	7.5	0.5	0.4886	0.0927	1.2237	4.9897	0.6233	0.1898
3	7.5	1.5	0.4346	0.0706	1.3166	5.6214	0.6114	0.1625
3	7.5	3.5	0.4156	0.0614	1.2740	5.5450	0.5963	0.1478
3	7.5	5.5	0.4101	0.0586	1.2437	5.4256	0.5904	0.1429
3	7.5	10.0	0.4056	0.0563	1.2104	5.2764	0.5850	0.1388
3	10.0	0.5	0.3321	0.0418	1.3159	5.5655	0.6157	0.1259
3	10.0	1.5	0.3115	0.0345	1.2730	5.5411	0.5961	0.1107
3	10.0	3.5	0.3051	0.0320	1.2197	5.3194	0.5864	0.1049
3	10.0	5.5	0.3032	0.0313	1.1991	5.2225	0.5833	0.1032
3	10.0	10.0	0.3018	0.0307	1.1805	5.1316	0.5807	0.1018

Table 4. Mean, variance, Skewness, Kurtosis, coefficient of variation and Index of dispersion for the GWR distribution

λ	θ	β	Mean	Variance	Skewness	Kurtosis	CV	γ
4	4.5	0.5	1.3721	0.3936	0.6806	3.6154	0.4572	0.2868
4	4.5	1.5	1.2006	0.3774	0.8501	3.8513	0.5117	0.3144
4	4.5	3.5	1.0711	0.3263	1.0264	4.3760	0.5333	0.3046
4	4.5	5.5	1.0177	0.2954	1.0901	4.6548	0.5341	0.2903
4	4.5	10.0	0.9665	0.2605	1.1231	4.8779	0.5281	0.2695
4	5.0	0.5	1.1945	0.3200	0.7120	3.6287	0.4735	0.2679
4	5.0	1.5	1.0341	0.2925	0.9192	4.0249	0.5230	0.2828
4	5.0	3.5	0.9291	0.2466	1.0744	4.5781	0.5345	0.2654
4	5.0	5.5	0.8892	0.2234	1.1145	4.7980	0.5315	0.2512
4	5.0	10.0	0.8526	0.1993	1.1208	4.9152	0.5236	0.2337
4	7.5	0.5	0.6784	0.1274	0.9448	4.0988	0.5262	0.1878
4	7.5	1.5	0.5971	0.1011	1.1097	4.7657	0.5324	0.1692
4	7.5	3.5	0.5634	0.0862	1.1159	4.9158	0.5212	0.1531
4	7.5	5.5	0.5530	0.0812	1.0954	4.8720	0.5153	0.1469
4	7.5	10.0	0.5444	0.0769	1.0649	4.7697	0.5094	0.1412
4	10.0	0.5	0.4581	0.0599	1.0898	4.6535	0.5341	0.1307
4	10.0	1.5	0.4222	0.0484	1.1153	4.9153	0.5210	0.1146
4	10.0	3.5	0.4099	0.0439	1.0741	4.8030	0.5110	0.1070
4	10.0	5.5	0.4064	0.0425	1.0529	4.7242	0.5074	0.1046
4	10	10	0.4036	0.0414	1.0320	4.6392	0.5043	0.1026

Table 5. Mean, variance, Skewness, Kurtosis, coefficient of variation and Index of dispersion for the GWR distribution

λ	θ	β	Mean	Variance	Skewness	Kurtosis	CV	γ
5	4.5	0.5	1.6589	0.4338	0.6277	3.5892	0.3970	0.2615
5	4.5	1.5	1.5149	0.4428	0.6972	3.5969	0.4392	0.2923
5	4.5	3.5	1.3758	0.4121	0.8369	3.8668	0.4666	0.2996
5	4.5	5.5	1.3080	0.3831	0.9111	4.0886	0.4732	0.2929
5	4.5	10.0	1.2360	0.3423	0.9755	4.3513	0.4734	0.2770
5	5.0	0.5	1.4624	0.3561	0.6329	3.5678	0.4081	0.2435
5	5.0	1.5	1.3170	0.3531	0.7441	3.6660	0.4512	0.2681
5	5.0	3.5	1.1946	0.3178	0.8904	4.0204	0.4719	0.2660
5	5.0	5.5	1.1404	0.2926	0.9511	4.2392	0.4743	0.2566
5	5.0	10.0	1.0863	0.2616	0.9900	4.4402	0.4708	0.2408
5	7.5	0.5	0.8662	0.1555	0.7638	3.7019	0.4552	0.1795
5	7.5	1.5	0.7663	0.1321	0.9419	4.2016	0.4743	0.1724
5	7.5	3.5	0.7165	0.1130	0.9926	4.4697	0.4691	0.1577
5	7.5	5.5	0.6999	0.1055	0.9870	4.4955	0.4641	0.1507
5	7.5	10.0	0.6856	0.0986	0.9646	4.4491	0.4581	0.1439
5	10.0	0.5	0.5887	0.0776	0.9108	4.0874	0.4732	0.1318
5	10.0	1.5	0.5368	0.0634	0.9927	4.4719	0.4690	0.1181
5	10.0	3.5	0.5170	0.0565	0.9724	4.4695	0.4598	0.1093
5	10.0	5.5	0.5110	0.0543	0.9536	4.4163	0.4559	0.1062
5	10.0	10.0	0.5062	0.0524	0.9320	4.3434	0.4524	0.1036

5.5 Harmonic Mean of the Generalized Weighted Rama Distribution

For the random variable X having the GWR distribution with parameters θ, β and λ, the harmonic mean is obtained as follows

$$\text{HM} = E\left(\frac{1}{X}\right) = \int_{-\infty}^{\infty} \left(\frac{1}{x}\right) f\left(x;\theta,\beta,\lambda\right) dx$$

$$= \frac{\theta^{\lambda+3}}{\left[\beta\theta^3 + \lambda(\lambda+1)(\lambda+2)\right]\Gamma(\lambda)} \int_0^{\infty} \left(\frac{1}{x}\right) x^{\lambda-1}\left(\beta + x^3\right) e^{-\theta x} dx$$

$$= \frac{\theta\left[\beta\theta^3 + \lambda(\lambda+1)(\lambda-1)\right]}{(\lambda-1)\left[\beta\theta^3 + \lambda(\lambda+1)(\lambda+2)\right]}, \lambda > 1 \tag{38}$$

5.6 Moment Generating Function and Characteristic Function of the Generalize Weighted Rama Distribution

For the random variable X having the GWR distribution with parameters θ, β and λ, the moment generating function (mgf) is obtained as follows

$$M_X(t) = E\left(e^{tX}\right) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E\left(X^r\right)$$

$$=\sum_{r=0}^{\infty}\frac{t^r}{r!}\frac{\Gamma(\lambda+r)}{\Gamma(\lambda)}\left[\frac{\beta\theta^3+(\lambda+r)(\lambda+r+1)(\lambda+r+2)}{\theta^r\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]}\right] \tag{39}$$

Similarly, the characteristic function (cf) of a random variable X having the GWR distribution with parameters θ, β and λ becomes

$$\phi_X(t)=E\left(e^{itX}\right)=\sum_{r=0}^{\infty}\frac{(it)^r}{r!}E\left(X^r\right)$$

$$=\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\frac{\Gamma(\lambda+r)}{\Gamma(\lambda)}\left[\frac{\beta\theta^3+(\lambda+r)(\lambda+r+1)(\lambda+r+2)}{\theta^r\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]}\right] \tag{40}$$

5.7 Bonferroni and Lorenz Curves for the Generalized Weighted Rama Distribution

The Bonferroni was developed by Bonferroni [36] and Lorenz curve by Lorenz [37]. The two curves can be used to measure the inequality of the distribution of a random variable. They are applicable in the field of economics but also in other areas like reliability, demography, insurance and medicine. Given that X is a nonnegative random variable with pdf $f(x)$, then the Bonferroni and Lorenz curves are defined as

$$B(p)=\frac{1}{p\mu}\int_0^q x\,f(x)\,dx$$

And

$$L(p)=\frac{1}{\mu}\int_0^q x\,f(x)\,dx$$

Respectively, where $\mu=E(X)$ and $q=F^{-1}(p)$.So, for the GWR distribution, one obtains the following Bonferroni curve

$$B(p)=\frac{1}{p\mu}\int_0^q x\frac{\theta^{\lambda+3}}{\beta\theta^3+\lambda(\lambda+1)(\lambda+2)}\frac{x^{\lambda-1}}{\Gamma(\lambda)}\left(\beta+x^3\right)e^{-\theta x}dx$$

$$\therefore B(p)=\frac{\beta\theta^3\gamma(\lambda+1,\theta q)+\gamma(\lambda+4,\theta q)}{p\lambda\Gamma(\lambda)\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]} \tag{41}$$

Similarly, the Lorenz curve of the GWR distribution is

$$L(p)=pB(p)=\frac{\beta\theta^3\gamma(\lambda+1,\theta q)+\gamma(\lambda+4,\theta q)}{\lambda\Gamma(\lambda)\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]} \tag{42}$$

where $\gamma(\lambda+1,\theta q)$ and $\gamma(\lambda+4,\theta q)$ are lower incomplete gamma functions.

5.8 Rényi Entropy of the Generalized Weighted Rama Distribution

Entropy is used to measure the randomness of systems and it is widely used in areas like physics, molecular imaging of tumours, sparse kernel density estimation, high-resolution scalar quantization, estimation of the number of components of a multi-component non-stationary signal, identification of cardiac autonomic neuropathy in diabetes and signal segmentation in time-frequency plane (Popescu and Aiordachioaie [38]). A large value of entropy implies that there is greater uncertainty in the data. The Rényi, Shannon and Tsallis entropy, among others, are some different forms of entropy. This paper concentrates on the Rényi entropy, which according to Rényi [39] is defined for the GWR distribution as follows

$$I_R(\eta)=\frac{1}{1-\eta}\log\left(\int_0^{\infty} f^{r}(x)dx\right),\eta>0 \text{ and } \eta\neq 1$$

$$=\frac{1}{1-\eta}\log\left\{\left(\frac{\theta^{\lambda+3}}{\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]\Gamma(\lambda)}\right)^{\eta}\int_0^{\infty}\left(\beta+x^3\right)^{\eta}x^{\eta(\lambda-1)}e^{-\eta\theta x}dx\right\}$$

$$=\frac{1}{1-\eta}\log\left(\int_0^{\infty}\frac{\theta^{\eta(\lambda+3)}}{\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]^{\eta}}\frac{x^{\eta(\lambda-1)}}{\Gamma^{\eta}(\lambda)}\left(\beta+x^3\right)^{\eta}e^{-\theta\eta x}dx\right)$$

$$=\frac{1}{1-\eta}\log\left(\int_0^{\infty}\frac{\theta^{\eta(\lambda+3)}}{\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]^{\eta}}\frac{x^{\eta(\lambda-1)}}{\Gamma^{\eta}(\lambda)}\sum_{j=0}^{\infty}\binom{\eta}{j}\beta^{\eta-j}x^{3j}e^{-\theta\eta x}dx\right)$$

$$=\frac{1}{1-\eta}\log\left(\frac{\theta^{\eta(\lambda+3)}}{\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]^{\eta}\Gamma^{\eta}(\lambda)}\sum_{j=0}^{\infty}\binom{\eta}{j}\beta^{\eta-j}\int_0^{\infty}x^{\eta(\lambda-1)+3j}e^{-\theta\eta x}dx\right)$$

$$I_R(\eta)=\frac{1}{1-\eta}\log\left(\sum_{j=0}^{\infty}\binom{\eta}{j}\frac{\beta^{\eta-j}\,\theta^{4\eta-3j-1}\,\Gamma(\eta(\lambda-1)+3j+1)}{\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]^{\eta}\Gamma^{\eta}(\lambda)\eta^{(\eta(\lambda-1)+3j+1)}}\right) \tag{43}$$

5.9 Distributions of Order Statistics for the Generalized Weighted Rama Distribution

Moments of order statistics play an important role in quality control testing and reliability to predict the failure items based on the times of fewer early failures. Let $X_1,X_2,...,X_n$ be a random sample of size n from the GWR distribution with cumulative distribution function (cdf) $F(x)$ and probability density function $f(x)$. Then, $X_{(1)},X_{(2)},...,X_{(n)}$ denote corresponding order statistics, where $X_{(1)}\leq X_{(2)},...\leq X_{(n)}$, $X_{(1)}=\min(X_1,X_2,...,X_n)$ and $X_{(n)}=\max(X_1,X_2,...,X_n)$. The probability density function (pdf) of the kth order statistic are given by

$$f_{X_{(k)}}(x)=\frac{n!}{(k-1)!(n-k)!}\left[F(x)\right]^{k-1}\left[1-F(x)\right]^{n-k}f(x)$$

$$=\frac{n!}{(k-1)!(n-k)!}\frac{\theta^{\lambda+3}}{\beta\theta^3+\lambda(\lambda+1)(\lambda+2)}\frac{x^{\lambda-1}}{\Gamma(\lambda)}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^{l}$$

$$\times\left\{1-\left(\frac{\begin{array}{l}\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]\Gamma(\lambda,\theta x)\\+\left[(\theta x)^2+(\lambda+2)(\theta x)+(\lambda+1)(\lambda+2)\right](\theta x)^\lambda e^{-\theta x}\end{array}}{\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]\Gamma(\lambda)}\right)\right\}^{k-1+l}\left(\beta+x^3\right)e^{-\theta x} \tag{44}$$

If $k=1$ in (44), one obtains the probability density function (pdf) of the minimum order statistic $X_{(1)}$ for the GWR distribution as

$$f_{X_{(1)}}(x)=\frac{n\theta^{\lambda+3}x^{\lambda-1}\left(\beta+x^3\right)e^{-\theta x}}{\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]\Gamma(\lambda)}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l\left\{1-\left(\frac{\begin{array}{l}\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]\Gamma(\lambda,\theta x)\\+\left[(\theta x)^2+(\lambda+2)(\theta x)+(\lambda+1)(\lambda+2)\right](\theta x)^\lambda e^{-\theta x}\end{array}}{\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]\Gamma(\lambda)}\right)\right\}^{n-1+l} \tag{45}$$

Similarly for $k=n$ in (44), we obtain the probability density function (pdf) of the maximum order statistic $X_{(n)}$ for the GWR distribution as

$$f_{X_{(n)}}(x)=\frac{n\theta^{\lambda+3}x^{\lambda-1}\left(\beta+x^3\right)e^{-\theta x}}{\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]\Gamma(\lambda)}\times\left\{1-\left(\frac{\begin{array}{l}\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]\Gamma(\lambda,\theta x)\\+\left[(\theta x)^2+(\lambda+2)(\theta x)+(\lambda+1)(\lambda+2)\right](\theta x)^\lambda e^{-\theta x}\end{array}}{\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]\Gamma(\lambda)}\right)\right\}^{n-1} \tag{46}$$

6 Reliability Measures of the Generalized Rama Distribution

In this section, we present the survival, hazard rate, reversed hazard, cumulative hazard and odds functions of the GWR distribution, which are very useful in reliability analysis.

6.1 Survival Function of the Generalized Weighted Rama Distribution

The survival function also known as the reliability function refers to the probability of surviving an age x or becoming older than age x. The survival function is useful in survival analysis and reliability theory. It is important in calculating systems' reliability. Given that X is a continuous random variable having the GWR distribution with parameters θ, β and λ, then the survival function of X is defined to be

$$S(x;\theta,\beta,\lambda)=1-F(x;\theta,\beta,\lambda)$$

$$S(x;\theta,\beta,\lambda)=\left[\frac{\left[\beta\theta^{3}+\lambda(\lambda+1)(\lambda+2)\right]\Gamma(\lambda,\theta x)+\left[(\theta x)^{2}+(\lambda+2)(\theta x)+(\lambda+1)(\lambda+2)\right](\theta x)^{\lambda}e^{-\theta x}}{\left[\beta\theta^{3}+\lambda(\lambda+1)(\lambda+2)\right]\Gamma(\lambda)}\right] \tag{47}$$

for $x>0,\theta>0,\beta,\lambda>0$.

6.2 Hazard Rate Function of the Generalized Weighted Rama Distribution

The hazard rate function of a statistical distribution is obtained mathematically as the ratio of the probability density function $f(x)$ to the survival function $S(x)$. Thus, the hazard rate function of the GWR distribution is defined as

$$h(x;\theta,\beta,\lambda)=\frac{f(x;\theta,\beta,\lambda)}{S(x;\theta,\beta,\lambda)}$$

$$h(x;\theta,\beta,\lambda)=\frac{\theta^{\lambda+3}x^{\lambda-1}\left(\beta+x^{3}\right)e^{-\theta x}}{\left[\beta\theta^{3}+\lambda(\lambda+1)(\lambda+2)\right]\Gamma(\lambda,\theta x)+\left[(\theta x)^{2}+(\lambda+2)(\theta x)+(\lambda+1)(\lambda+2)\right](\theta x)^{\lambda}e^{-\theta x}} \tag{48}$$

for $x>0,\theta>0,\beta,\lambda>0$.

For the GWR distribution, the behaviour of $S(x)$ and $h(x)$ for different values of the parameters θ, β and λ are shown in Figs. 3 and 4 respectively.

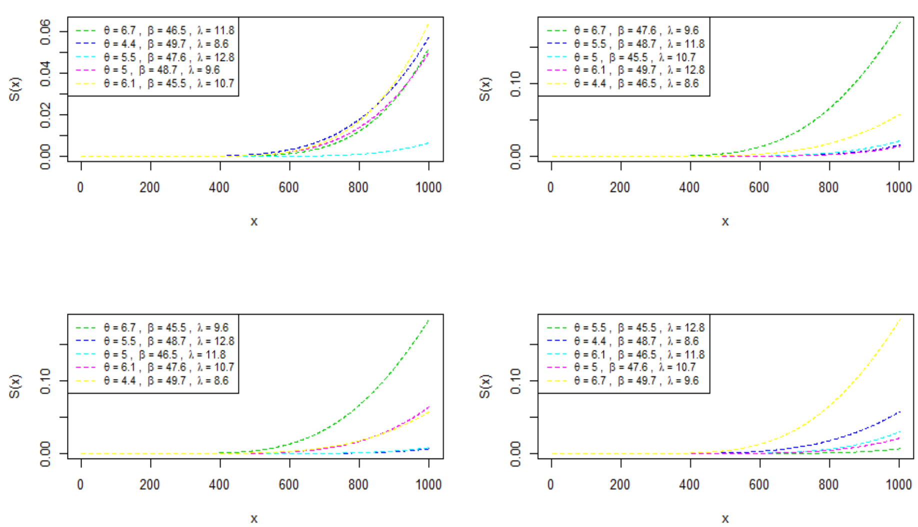

Fig. 3. Survival function shapes for the GWR distribution considering different values of θ, β and λ

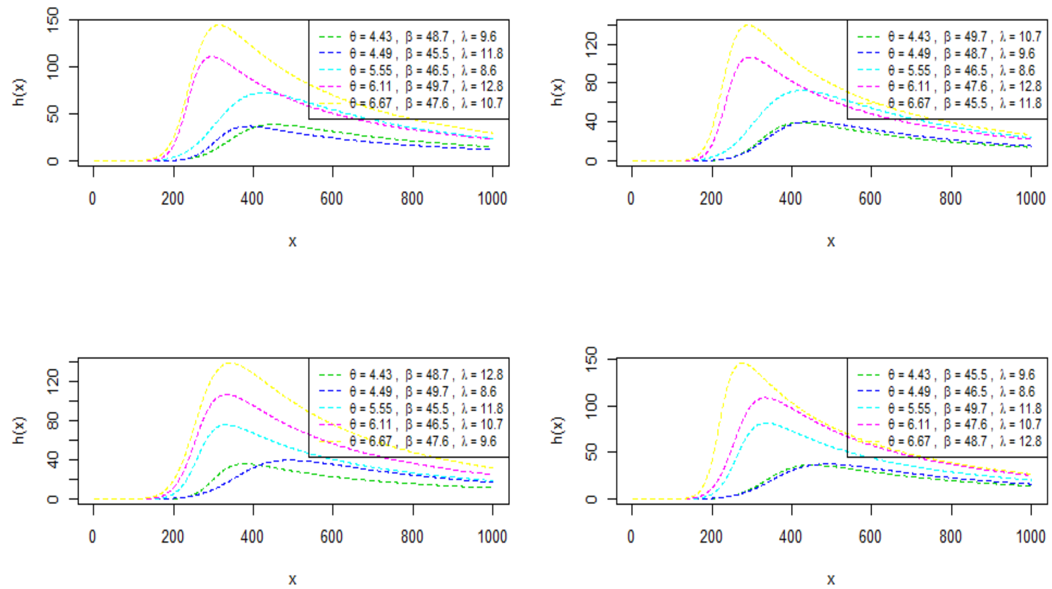

Fig. 4. Hazard function shapes of for GWR distribution considering different values of θ, β and λ

6.3 Reversed Hazard Rate Function of the Generalized Weighted Rama Distribution

The reversed hazard rate refers to the ratio of the probability density function (pdf) to the cumulative distribution function (cdf). It extends the concept of hazard rate to a reverse time direction and is given by

$$h_R(x;\theta,\beta,\lambda)=\frac{f(x;\theta,\beta,\lambda)}{F(x;\theta,\beta,\lambda)}$$

$$h_R(x;\theta,\beta,\lambda)=\frac{\dfrac{\theta^{\lambda+3}}{\beta\theta^3+\lambda(\lambda+1)(\lambda+2)}\dfrac{x^{\lambda-1}}{\Gamma(\lambda)}\left(\beta+x^3\right)e^{-\theta x}}{1-\left[\dfrac{\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]\Gamma(\lambda,\theta x)+\left[(\theta x)^2+(\lambda+2)(\theta x)+(\lambda+1)(\lambda+2)\right](\theta x)^{\lambda}e^{-\theta x}}{\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]\Gamma(\lambda)}\right]} \quad (49)$$

for $x>0,\theta>0,\beta,\lambda>0$.

The reversed hazard $h_R(x;\theta,\beta,\lambda)$ describes the probability of an immediate past failure, given that the unit has already failed at time x, described by $h(x;\theta,\beta,\lambda)$.

6.4 Cumulative Hazard Rate Function of the Generalized Weighted Rama Distribution

The cumulative hazard rate function of the GWR distribution is given by

$$H(x;\theta,\beta,\lambda) = -\ln\left[1 - F(x;\theta,\beta,\lambda)\right]$$

$$H(x;\theta,\beta,\lambda) = -\ln\left\{\left[\frac{\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]\Gamma(\lambda,\theta x)+\left[(\theta x)^2+(\lambda+2)(\theta x)+(\lambda+1)(\lambda+2)\right](\theta x)^\lambda e^{-\theta x}}{\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]\Gamma(\lambda)}\right]\right\} \quad (50)$$

Notice that $H(x;\theta,\beta,\lambda)$ does not have a probabilistic connotation although it plays a key role in reliability and survivals analysis since $P(X > x) = e^{-H(x)}$, $x \geq 0$.

6.5 Odds Function for the Generalized Weighted Rama Distribution

The odds function of the GWR distribution is defined as

$$O(x;\theta,\beta,\lambda) = \frac{F(x;\theta,\beta,\lambda)}{1-(x;\theta,\beta,\lambda)}$$

$$O(x;\theta,\beta,\lambda) = \frac{1-\left[\frac{\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]\Gamma(\lambda,\theta x)+\left[(\theta x)^2+(\lambda+2)(\theta x)+(\lambda+1)(\lambda+2)\right](\theta x)^\lambda e^{-\theta x}}{\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]\Gamma(\lambda)}\right]}{\left[\frac{\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]\Gamma(\lambda,\theta x)+\left[(\theta x)^2+(\lambda+2)(\theta x)+(\lambda+1)(\lambda+2)\right](\theta x)^\lambda e^{-\theta x}}{\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]\Gamma(\lambda)}\right]} \quad (51)$$

6.6 Mean Residual Life Function of the Generalized Weighted Rama Distribution

The mean residual life function is the expected additional lifetime $(X - x)$, given that the item has survived to time x. Thus, in life testing situations, the expected additional lifetime given that a component has survived until time x is called the mean residual life. The mean residual life function, say, $m(x)$ is given by

$$m(x) = E(X - x | X > x) = \frac{1}{P(X > x)}\int_x^\infty P(X > t)\,dt = \frac{1}{S(x)}\left(\int_x^\infty t\,f(t)\,dt\right) - x\ , t \geq 0 \quad (52)$$

where

$$\int_x^\infty t\,f(t)\,dt = \frac{\theta^{\lambda+3}}{\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]\Gamma(\lambda)}\left[\beta\int_t^\infty t^{(\lambda+1)-1}e^{-\theta t} + \int_t^\infty t^{(\lambda+4)-1}e^{-\theta t}\right]$$

$$= \frac{\beta\theta^3\Gamma(\lambda+1,\theta x)+\Gamma(\lambda+4,\theta x)}{\theta\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]\Gamma(\lambda)} \quad (53)$$

But from upper incomplete gamma function, it is known that $\Gamma(s+1,z) = s\,\Gamma(s,z) + z^s e^{-z}$. Consequently,

$$\Gamma(\lambda+4,\theta x)=\lambda(\lambda+1)(\lambda+2)(\lambda+3)\Gamma(\lambda,\theta x) \\ +\left[(\lambda+1)(\lambda+2)(\lambda+3)+(\lambda+2)(\lambda+3)(\theta x)+(\lambda+3)(\theta x)^2+(\theta x)^3\right](\theta x)^\lambda e^{-\theta x} \tag{54}$$

Substituting (13), (47), (53) and (54) into (52), we obtain

$$m(x)=\frac{\left[\lambda\beta\theta^3+\lambda(\lambda+1)(\lambda+2)(\lambda+3)\right]\Gamma(\lambda,\theta x)+\begin{bmatrix}(\lambda+1)(\lambda+2)(\lambda+3)+(\lambda+2)(\lambda+3)(\theta x)\\+(\lambda+3)(\theta x)^2+(\theta x)^3+\beta\theta^3\end{bmatrix}(\theta x)^\lambda e^{-\theta x}}{\theta\left\{(\theta x)^\lambda\left[(\theta x)^2+(\lambda+2)(\theta x)+(\lambda+1)(\lambda+2)\right]e^{-\theta x}+\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]\Gamma(\lambda,\theta x)\right\}}-x$$

$$m(x)=\frac{\begin{Bmatrix}\left[\lambda\left(\beta\theta^3+(\lambda+1)(\lambda+2)(\lambda+3)\right)-\theta x\left\{\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right\}\right]\Gamma(\lambda,\theta x)\\+(\theta x)^\lambda\left\{(\theta x)^2+2(\lambda+2)(\theta x)+\beta\theta^3+(\lambda+1)(\lambda+2)(\lambda+3)(\theta x)^\lambda\right\}e^{-\theta x}\end{Bmatrix}}{\theta\left[(\theta x)^\lambda\left\{(\theta x)^2+(\lambda+2)(\theta x)+(\lambda+1)(\lambda+2)\right\}e^{-\theta x}+\left\{\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right\}\Gamma(\lambda,\theta x)\right]} \tag{55}$$

7 Maximum Likelihood Estimators of the Generalized Weighted Rama Distribution

The method of maximum likelihood is the most frequently used method of estimation of parameters of statistical distributions (Casella and Berger [40]). Undoubtedly, the success of the maximum likelihood method is due to the fact that its estimators' possess desirable properties such as consistency, asymptotic efficiency and invariance property. Thus, to obtain the maximum likelihood estimators of parameters of the GWR distribution, we let $X_1, X_2, ..., X_n$ denote a random sample of size n from this distribution and define the likelihood function of the random sample as

$$L=L(\theta,\beta,\lambda|x)=\prod_{i=1}^{n}\frac{\theta^{\lambda+3}}{\beta\theta^3+\lambda(\lambda+1)(\lambda+2)}\frac{x_i^{\lambda-1}}{\Gamma(\lambda)}\left(\beta+x_i^3\right)e^{-\theta x_i}$$

$$=\left(\frac{\theta^{\lambda+3}}{\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]\Gamma(\lambda)}\right)^n e^{-\theta\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}x_i^{\lambda-1}\left(\beta+x_i^3\right) \tag{56}$$

Taking the natural log of (56) gives the log-likelihood function of the GWR distribution as

$$\ln L=n\left[(\lambda+3)\ln(\theta)-\ln\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]-\ln\Gamma(\lambda)\right]+(\lambda-1)\sum_{i=1}^{n}\ln x_i+\sum_{i=1}^{n}\ln\left(\beta+x_i^3\right)-\theta\sum_{i=1}^{n}x_i \tag{57}$$

The maximum likelihood estimates $(\hat{\theta},\hat{\beta},\hat{\lambda})$ of the unknown parameters (θ,β,λ) of GWR distribution are the solution of the following nonlinear system of equations:

$$\frac{\partial \ln L}{\partial \theta}=\frac{n(\lambda+3)}{\theta}-\frac{3n\beta\theta^2}{\beta\theta^3+\lambda(\lambda+1)(\lambda+2)}-\sum_{i=1}^{n}x_i=0 \tag{58}$$

$$\frac{\partial \ln L}{\partial \beta}=-\frac{n\theta^3}{\beta\theta^3+\lambda(\lambda+1)(\lambda+2)}-\sum_{i=1}^{n}\frac{1}{\beta+x_i^3}=0 \tag{59}$$

$$\frac{\partial \ln L}{\partial \lambda}=n\ln(\theta)-\frac{n\left(3\lambda^2+6\lambda+2\right)}{\beta\theta^3+\lambda(\lambda+1)(\lambda+2)}-n\,\psi(\lambda)+\sum_{i=1}^{n}\ln(x_i)=0 \tag{60}$$

where $\psi(\lambda)=\frac{d}{d\lambda}\ln\Gamma(\lambda)=\frac{\Gamma'(\lambda)}{\Gamma(\lambda)}$ is the digamma function of λ.

It is usually more convenient to use nonlinear optimization algorithms such as quasi-Newton algorithm to numerically maximize the log-likelihood function since the system of equations (58), (59) and (60) cannot yield closed form solution. The R package provides nonlinear optimization for solving such problems.

8 Asymptotic Confidence Intervals of the MLEs of The Generalized Weighted Rama Distribution

Since the maximum likelihood estimators of the unknown parameters θ, β and λ cannot be derived in closed form, it is not easy to derive the exact distributions of the maximum likelihood estimators. Hence, the exact confidence intervals for the parameter cannot be obtained. It is customary to use the large sample approximation to derive the asymptotic distribution of the maximum likelihood estimators. Consequently, the asymptotic distribution of the MLEs is

$$\left[\sqrt{n}\left(\hat{\theta}_{MLE}-\theta\right),\sqrt{n}\left(\hat{\beta}_{MLE}-\beta\right),\sqrt{n}\left(\hat{\lambda}_{MLE}-\lambda\right)\right]\xrightarrow{d}N_3\left(0,I^{-1}(\theta,\beta,\lambda)\right)$$

where $\boldsymbol{I}(\theta,\beta,\lambda)$ the Fisher information matrix of the unknown parameters θ,β and λ is defined as

$$\boldsymbol{I}(\theta,\beta,\lambda)=\begin{bmatrix}-E\left(\frac{\partial^2\ln L(\theta,\beta,\lambda)}{\partial\theta^2}\right) & -E\left(\frac{\partial^2\ln L(\theta,\beta,\lambda)}{\partial\theta\,\partial\beta}\right) & -E\left(\frac{\partial^2\ln L(\theta,\beta,\lambda)}{\partial\theta\,\partial\lambda}\right)\\ -E\left(\frac{\partial^2\ln L(\theta,\beta,\lambda)}{\partial\beta\,\partial\theta}\right) & -E\left(\frac{\partial^2\ln L(\theta,\beta,\lambda)}{\partial\theta^2}\right) & -E\left(\frac{\partial^2\ln L(\theta,\beta,\lambda)}{\partial\beta\,\partial\lambda}\right)\\ -E\left(\frac{\partial^2\ln L(\theta,\beta,\lambda)}{\partial\lambda\,\partial\theta}\right) & -E\left(\frac{\partial^2\ln L(\theta,\beta,\lambda)}{\partial\lambda\,\partial\beta}\right) & -E\left(\frac{\partial^2\ln L(\theta,\beta,\lambda)}{\partial\lambda^2}\right)\end{bmatrix}_{(\theta,\beta,\lambda)=\left(\hat{\theta},\hat{\beta},\hat{\lambda}\right)} \tag{61}$$

Notably, the variance-covariance matrix of the GWR distribution is given by

$$\boldsymbol{I}^{-1}(\theta,\beta,\lambda)=\begin{pmatrix}\mathrm{var}\left(\hat{\theta}\right) & \mathrm{cov}\left(\hat{\theta},\hat{\beta}\right) & c\,\mathrm{ov}\left(\hat{\theta},\hat{\lambda}\right)\\ \mathrm{cov}\left(\hat{\beta},\hat{\theta}\right) & \mathrm{var}\left(\hat{\beta}\right) & \mathrm{cov}\left(\hat{\beta},\hat{\lambda}\right)\\ \mathrm{cov}\left(\hat{\lambda},\hat{\theta}\right) & \mathrm{cov}\left(\hat{\lambda},\hat{\beta}\right) & \mathrm{var}\left(\hat{\lambda}\right)\end{pmatrix} \tag{62}$$

and the elements of the Fisher information matrix $\boldsymbol{I}(\theta,\beta,\lambda)$ are defined as follows

$$\frac{\partial^2 \ln L(\theta,\beta,\lambda)}{\partial\theta^2}=\frac{9n\beta^2\theta^2-6n\beta\theta\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]}{\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]^2}-\frac{n(\lambda+3)}{\theta^2} \tag{63}$$

$$\frac{\partial^2 \ln L(\theta,\beta,\lambda)}{\partial\beta^2}=\frac{n\theta^6}{\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]^2}-\sum_{i=1}^{n}\frac{1}{\left(\beta+x_i^3\right)^2} \tag{64}$$

$$\frac{\partial^2 \ln L(\theta,\beta,\lambda)}{\partial\lambda^2}=\frac{n\left[3\lambda^4+12\lambda^3-18\lambda^2+48\lambda-6(\lambda+1)\beta\theta^3+4\right]}{\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]^2}-n\,\psi^1(\lambda) \tag{65}$$

$$\frac{\partial^2 \ln L(\theta,\beta,\lambda)}{\partial\theta\,\partial\beta}=\frac{3n\theta^2\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)-\beta\theta^3\right]}{\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]^2}=\frac{\partial^2 \ln L(\theta,\beta,\lambda)}{\partial\beta\,\partial\theta} \tag{66}$$

$$\frac{\partial^2 \ln L(\theta,\beta,\lambda)}{\partial\theta\,\partial\lambda}=\frac{n}{\theta}+\frac{3n\beta\theta^2\left[\lambda(\lambda+1)(\lambda+2)\right]}{\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]^2}=\frac{\partial^2 \ln L(\theta,\beta,\lambda)}{\partial\lambda\,\partial\theta} \tag{67}$$

$$\frac{\partial^2 \ln L(\theta,\beta,\lambda)}{\partial\lambda\,\partial\beta}=\frac{n\theta^3\left(3\lambda^2+6\lambda+2\right)}{\left[\beta\theta^3+\lambda(\lambda+1)(\lambda+2)\right]^2}=\frac{\partial^2 \ln L(\theta,\beta,\lambda)}{\partial\beta\,\partial\lambda} \tag{68}$$

Now, the approximate $100(1-\tau)\%$ confidence intervals of the parameters θ, β and λ of the GWR distribution takes the forms

$$\hat{\theta}\pm Z_{\tau/2}\sqrt{\operatorname{var}\left(\hat{\theta}\right)} \tag{69}$$

$$\hat{\beta}\pm Z_{\tau/2}\sqrt{\operatorname{var}\left(\hat{\beta}\right)} \tag{70}$$

and

$$\hat{\lambda}\pm Z_{\tau/2}\sqrt{\operatorname{var}\left(\hat{\lambda}\right)} \tag{71}$$

where $Z_{\tau/2}$ is the upper $(\tau/2)$ percentile of the standard normal distribution.

9 Monte Carlo Simulation Study

This section deals with the simulation study for the assessment of the performance of the maximum likelihood estimates of the GWR distribution. Equation (15) was used to generate variates of the GWR distribution upon which the maximum likelihood estimates were obtained. In addition, the average bias and mean square errors were obtained and displayed in Table 6 using the formula $Bias\left(\hat{\Theta}\right)=N^{-1}\sum_{i=1}^{N}\left(\hat{\Theta}_i-\Theta\right)$ and $MSE\left(\hat{\Theta}\right)=N^{-1}\sum_{i=1}^{N}\left(\hat{\Theta}_i-\Theta\right)^2$, where $\Theta=\left(\theta,\beta,\lambda\right)$. Simulation results were obtained for different combinations of θ, β and λ. Clearly, the results in Table 6 show that the estimate are reasonably consistent and approaches the true parameter values. Consequently, it is concluded that the maximum likelihood technique performs well in estimation of parameters of the GWR distribution.

Table 6. Average values of MLE, corresponding MSE and Bias

	Set 1: $\theta=0.4$, $\beta=0.5$, $\lambda=0.5$				**95% approximate confidence interval (CI)**	
n	Parameters	MLE	SE	Biases	95% Lower CI	95% Upper CI
50	$\hat{\theta}$	0.4114	0.0623	0.0114	0.2893	0.5335
	$\hat{\beta}$	0.7852	1.7147	02852	-2.5756	4.1460
	$\hat{\lambda}$	0.5067	0.5019	0.0067	- 0.4770	1.4904
100	$\hat{\theta}$	0.4251	0.0600	0.0251	0.3075	0.5427
	$\hat{\beta}$	2.3680	4.7580	1.8680	-6.9577	11.6937
	$\hat{\lambda}$	0.7811	0.5942	0.2811	-0.3835	1.9457
	Set 2: $\theta=0.5$, $\beta=0.5$, $\lambda=0.5$					
50	$\hat{\theta}$	0.4987	0.0610	0.0013	0.3791	0.6183
	$\hat{\beta}$	0.3195	0.6229	0.1805	-0.9014	1.5404
	$\hat{\lambda}$	0.3667	0.3543	0.1333	-0.3277	1.0611
100	$\hat{\theta}$	0.5121	0.0500	0.0121	0.4141	0.6101
	$\hat{\beta}$	1.0311	1.3301	0.5311	-1.5759	3.6381
	$\hat{\lambda}$	0.5771	0.3325	0.0771	-0.0746	1.2288
	Set 3: $\theta=0.6$, $\beta=0.5$, $\lambda=0.5$					
50	$\hat{\theta}$	0.6039	0.0657	0.0039	0.4751	0.7327
	$\hat{\beta}$	0.3581	0.5564	0.1419	-0.7324	1.4486
	$\hat{\lambda}$	0.3960	0.3001	0.1040	-0.1922	0.9842
100	$\hat{\theta}$	0.6090	0.0557	0.0090	0.4998	0.7182
	$\hat{\beta}$	1.6027	1.6169	1.1027	-1.5660	4.7718
	$\hat{\lambda}$	0.7769	0.3300	0.2769	0.1301	1.4237

10 Application of Generalized Weighted Rama Distribution

In this section, we use a real data to demonstrate the usefulness of the GWR distribution. The data set which was reported in (Gross and Clark [41]) relates to relief (in minutes) of receiving analgesic of 20 patients. The data are given below:

1.1,	1.4,	1.3,	1.7,	1.9,	1.8,	1.6,	2.2,	1.7,	2.7,
4.1,	1.8,	1.5,	1.2,	1.4,	3.0,	1.7,	2.3,	1.6,	2.0

The GWR distribution is fitted to the data set by using the method of maximum likelihood and the results are compared with the following competitive models:

Two parameter Rama distribution (Umeh et al. [31]) with pdf:

$$f(x;\theta,\beta)=\frac{\theta^4}{\beta\theta^3+6}\left(\beta+x^3\right)e^{-\theta x};x>0,\theta>0,\beta>0$$

Two-parameter weighted Rama distribution (Eyob and Shanker [20]) with pdf:

$$f(x;\theta,\alpha)=\frac{\theta^{\alpha+3}}{\theta^3+\alpha(\alpha+1)(\alpha+2)}\frac{x^{\alpha-1}}{\Gamma(\alpha)}\left(1+x^3\right)e^{-\theta x};x>0,\theta>0,\ \alpha>0$$

Weighted Two-parametric Rama distribution (Vijayakumar et al. [28]) with pdf:

$$f(x;\theta,\alpha,c)=\frac{\theta^{c+4}\,x^c\left(\alpha+x^3\right)e^{-\theta x}}{\alpha\,\theta^3\,\Gamma(c+1)+\Gamma(c+4)},x>0,\theta>0,\alpha>0,c>0$$

Next, some criteria like the Akaike information criterion (AIC), Bayesian information criterion (BIC), and Akaike information criterion corrected (AICC) are used to verify which of the aforementioned distributions fits the research data better. The formulae for (AIC), (BIC) and (AICC) are respectively given by

$$AIC=2k-2l \tag{72}$$

$$BIC=k\ln(n)-2l \tag{73}$$

$$AICC=AIC+\frac{2k(k+1)}{n-k-1} \tag{74}$$

where l denotes the log-likelihood function evaluated at the maximum likelihood estimates, k is the number of model parameters, n is the sample size. For calculation of the analytical measures, the optimum () R-function with the argument method= "BFGS" in R (R Development Core Team [42])

A distribution is said to provide the best fit to the data if among all the distributions under consideration, it corresponds to minimum values of AIC, AICC, BIC and the log-likelihood respectively. The maximum likelihood estimates with the standard error of the fitted models and the 95 percent approximate confidence intervals are presented in Table 7. Also, the corresponding model selection criteria for the data set are presented in Table 8. Based on the results displayed in Table 8 respectively, it is evident that the GWRD has the smallest AIC, AICC, BIC and log-likelihood values among all competing models, and so it could be chosen as the best model among all the distributions which have been fitted to the real data set.

Table 7. Maximum likelihood estimates of parameters, standard errors and approximate confidence intervals

		Standard	95% Approximate Confidence Interval	
Distribution	**MLE**	**Error**	**Lower CI**	**Upper CI**
GWRD	$\hat{\theta} = 5.5819$	1.8111	2.0321	9.1317
	$\hat{\beta} = 47.9980$	123.2140	-193.5010	289.4974
	$\hat{\lambda} = 10.1780$	3.0937	4.1143	16.2417
WTPRD	$\hat{\theta} = 1.7390$	0.19237	1.3620	2.1160
	$\hat{\alpha} = 1.3147$	0.1593	1.0025	1.6269
	$\hat{c} = 1.2153$	0.1384	0.9440	1.4866
TPRD	$\hat{\theta} = 1.6331$	0.7723	0.1193	3.1468
	$\hat{\alpha} = 0.5212$	0.2655	0.0008	1.0416

Table 8. Model selection criteria

Model	Log-lik	AIC	AICC	BIC
GWRD	**17.7119**	**41.4238**	**42.9238**	**44.4110**
WTPRD	21.1572	48.4257	48.4753	53.3258
TPRD	28.3827	59.5643	59.7934	72.5474

11 Conclusion

In this paper, we introduced a new three-parameter Rama distribution called the generalized weighted Rama (GWR) distribution and derived some of its properties like moments, mean, variance, coefficient of variation, skewness, kurtosis, index of dispersion, harmonic mean, moment generating function, characteristic function, Bonferroni and Lorenz curves, quantile function, Rényi entropy, stochastic ordering and the pdf of order statistics. In addition, some functions commonly used in reliability analysis, such as survival, hazard, reversed, cumulative hazard, mean residual life and odds functions respectively have been derived. The parameters of the GWR distribution were estimated by using the maximum likelihood estimation procedure and the asymptotic confidence intervals were estimated. The stability of the maximum likelihood estimates of model parameters was assessed through a simulation. Finally, the GWR distribution was fitted to a real-life data set and was compared with estimates from some extensions of the Rama distribution. The GWR distribution was found to provide a better fit than the competing distributions considered in this study. It is hoped that the GWR distribution will serve as an alternative model for modelling data sets exhibiting positive skewness and upside-down bathtub shape hazard rate.

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Competing Interests

Authors have declared that no competing interests exist.

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