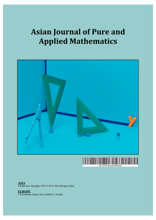

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A Study on Properties of skew (n, m) Binormal Operators

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Authors' contributions

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Abstract

In this paper, the class of skew (n, m)-binormal operators acting on a Hilbert space (H) is introduced. An operator $T \in B(H)$ is skew (n, m) binormal operators if it satisfies the condition $(T^{*m}T^nT^nT^{*m})T = T(T^nT^{*m}T^{*m}T^n)$. We investigate some of the basic properties of this class of operators. In particular, it has been shown that any scalar multiple of a skew (n, m) binormal operator is also skew (n, m) binormal. A counter example is provided to show that the class of (n, m) binormal operators is not in general contained in the class of skew (n, m) binormal operators. The concept of (n, m)-unitary quasiequivalence is introduced and shown to be an equivalence relation. It is further shown that if an operator T is skew (n, m)-binormal, and is unitarily equivalent to an operator S, then S is also skew (n, m)-binormal.

Keywords: Skew-(n, m)-binormal; isometric equivalence; (n, m)-unitary equivalence.

2020 Mathematics Subject Classification: 47A63, 47B02.

1 Introduction

This section provides definitions and basic results that are required.

Throughout this paper $B(H)$ denotes the algebra of bounded linear operators in a complex Hilbert Space H. The study of binormal operators was initiated by Campbell in [1]. Campbell realized while working on his second paper in [2], that the term binormal had already been used, but with a different definition. Brown [3] had defined a binormal operator as essentially being made up of 2×2 commuting normal matrices acting in the

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usual fashion in a Hilbert space of the form $\mathbb{H} \oplus \mathbb{H}$. Campbell [2] stated that usage of term was not current, so he continued his use of the term binormal to describe the definition we use in this paper. Unfortunately, both definitions are still used in Garcia and Wogen [4] and Jung, Kim, and Ko [5]. Classes of operators related to normal operators were introduced and studies by several researchers. Jibril [6] introduced the class of n-normal operators that an operator, where an operator $T \in B(H)$ is n-power normal if $T^nT^* = T^*T^n$. He gave some properties of these operators in general, and also study the special case when an operator is n-power normal for $n = 2, 3$. The class of (n, m)-normal operators was introduced by Abood and Al-loz [7]. An operator $T \in B(H)$ is (n, m)-normal if $[T^n, T^{*m}] = 0$, quasinormal if $[T, T^*T] = 0$, n-quasinormal if $[T, T^*T^n] = 0$, skew normal if $(T^*T)T = T(TT^*)$, binormal if $[TT^*, T^*T] = 0$, and n-binormal if $[T^nT^*, T^*T^n] = 0$.

Definition 1.1. [8] Let $T \in B(H)$. Then T is (n, m)-binormal if $T^{*m}T^nT^nT^{*m} = T^nT^{*m}T^{*m}T^n$ for positive integers m and n.

Definition 1.2. Let $T \in B(H)$. Then T is skew (n, m)-binormal if $(T^{*m}T^nT^nT^{*m})T = T(T^nT^{*m}T^{*m}T^n)$ for positive integers m and n.

Definition 1.3. Let $T \in B(H)$. Then T is k-skew (n, m)-binormal if $(T^{*m}T^nT^nT^{*m})T^k = T^k(T^nT^{*m}T^{*m}T^n)$ for positive integers k, m and n.

Remark. *For an operator T that is skew (n, m) binormal. We see that, If $k = m = n = 1$, then skew (n, m)-binormal becomes skew binormal.*

In Meenambiku, Seshaiah, and Sivamani [9], it is indicated that the class of binormal operators is contained in the class of skew n-binormal operators, that is every binormal operator is skew binormal. In this paper, we present a counter example to indicate that in general the class of binormal operators is not contained in the class of skew binormal and consequently it does not necessarily belong to the class of k-skew binormal operators [10, 11].

Example 1.1. *Consider the matrix*

$$T = \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \textit{then} \qquad T^* = \begin{bmatrix} 0 & 0 & 0 \\ -i & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

We now check if the matrix T is binormal

$$TT^*T^*T = \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -i & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -i & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$T^*TTT^* = \begin{bmatrix} 0 & 0 & 0 \\ -i & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -i & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We have

$$TT^*T^*T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = T^*TTT^*$$

Hence T is a binormal operator.

Let us now check if T is skew binormal.

$$(T^*TTT^*)T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$T(TT^*T^*T) = \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We have

$$(T^*TTT^*)T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = T(TT^*T^*T)$$

Therefore T is not skew binormal.

We further show that every skew binormal operator is not k skew binormal. Simple computation gives

$$T^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Considering the case when $k = 3$, we check if T is 3-skew binormal.

$$(T^*TTT^*)T^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = T^3(TT^*T^*T)$$

Hence T is 3-skew binormal.

We have therefore shown that the inclusion

$$binormal \subset skew\ binormal \subset k - skew\ binormal$$

does not always hold.

2 Main Results

In this section, we present results on the equivalences of skew(n, m)-binormal operators.

Proposition 2.1. *The relation (n, m)-unitary quasiequivalence is an equivalence relation.*

Proof. We show that the three conditions that define an equivalence relations are satisfied by the class of (n, m)-unitary quasiequivalent operators in Hilbert space.
Let A,B, and C be operators that are bounded in a Hilbert space (H) and U be a unitary operator. Clearly, A is (n, m)-unitary quasiequivalent to A since $A^{m*}A^n = IA^{m*}A^nI^*$ for $I = U$.
Let A be (n, m)-unitary quasiequivalent to B, then

$$A^{m*}A^n = UB^{m*}B^nU^*$$

and

$$A^nA^{m*} = UB^nB^{m*}U^*$$

Premultiplying and postmultiplying the preceding equation by U^* and U respectively yields

$$B^{m*}B^n = UA^{m*}A^nU^*$$

and

$$B^nB^{m*} = UA^nA^{m*}U^*$$

Hence B is (n, m)-unitary quasiequivalent to A. Lastly, suppose A is (n, m)-unitary quasiequivalent to B and B is (n, m)-unitary quasiequivalent to C, then

$$A^{m*}A^n = UB^{m*}B^nU^*$$

and

$$A^nA^{m*} = UB^nB^{m*}U^*$$

and

$$B^{m*}B^n = VA^{m*}A^nV^*$$

and

$$B^nB^{m*} = VA^nA^{m*}V^*$$

for U and V unitary operators. Then

$$A^nA^{m*} = UB^nB^{m*}U^* = UVC^nC^{m*}V^*U^* = WC^nC^{m*}W^*$$

where $W = UV$ is unitary. Also,

$$A^{m*}A^n = UB^{m*}B^nU^* = UVC^{m*}C^nV^*U^* = WC^{m*}C^nW^*$$

Thus, A is (n, m)-unitary quasiequivalent to C. Therefore, (n, m)-unitary quasiequivalence is an equivalence relation. □

Remark. *Let $T \in B(H)$ be skew (n, m) binormal operator, then βT is skew (n, m) binormal for every real scalar β*

Proof. Since $T \in B(H)$ is skew (n, m) binormal operator, it follows that

$$(T^{*m}T^nT^nT^{*m})T = T(T^nT^{*m}T^{*m}T^n)$$

We show that βT is skew (n, m) binormal.

$$\begin{aligned}((\beta T)^{*m}(\beta T)^n(\beta T)^n(\beta T)^{*m})(\beta T) &= (\beta^{*m}T^{*m}\beta^nT^n\beta^nT^n\beta^{*m}T^{*m})(\beta T)\\ &= \beta^{*m}\beta^n\beta^n\beta^{*m}\beta((T^{*m}T^nT^nT^{*m})T)\\ &= \beta^{*m}\beta^n\beta^n\beta^{*m}\beta(T(T^nT^{*m}T^{*m}T^n))\\ &= \beta T(\beta^nT^n\beta^{*m}T^{*m}\beta^{*m}T^{*m}\beta^nT^n)\\ &= (\beta T)((\beta T)^n(\beta T)^{*m}(\beta T)^{*m}(\beta T)^n)\end{aligned}$$

Hence βT is skew (n, m) binormal operator. □

Remark. *Let $T \in B(H)$ be skew (n, m) binormal operator, then T^* is also skew (n, m) binormal.*

Proof. Since $T \in B(H)$ is skew (n, m) binormal operator, it follows that

$$(T^{*m}T^nT^nT^{*m})T = T(T^nT^{*m}T^{*m}T^n)$$

Taking the adjoint on both sides,

$$((T^{*m}T^nT^nT^{*m})T)^* = (T(T^nT^{*m}T^{*m}T^n))^*$$

$$(T)^*\left[(T^{*m})^*(T^n)^*(T^n)^*(T^{*m})^*\right] = \left[(T^n)^*(T^{m*})^*(T^{m*})^*(T^n)^*\right](T)^*$$

$$(T^*)\left[(T^*)^{*m}(T^*)^n(T^*)^n(T^*)^{*m}\right] = \left[(T^*)^n(T^*)^{*m}(T^*)^{*m}(T^*)^n\right](T^*)$$

Therefore T^* is also skew (n, m) binormal □

Remark. *Let $T \in B(H)$ be skew (n, m) binormal operator, then T^{-1} is also skew (n, m) binormal.*

Proof. Since $T \in B(H)$ is skew (n,m) binormal operator, it follows that

$$(T^{*m}T^nT^nT^{*m})T = T(T^nT^{*m}T^{*m}T^n)$$

Taking the inverse on both sides,

$$((T^{*m}T^nT^nT^{*m})T)^{-1} = (T(T^nT^{*m}T^{*m}T^n))^{-1}$$
$$(T)^{-1}\left[(T^{*m})^{-1}(T^n)^{-1}(T^n)^{-1}(T^{*m})^{-1}\right] = \left[(T^n)^{-1}(T^{m*})^{-1}(T^{m*})^{-1}(T^n)^{-1}\right](T)^{-1}$$
$$(T^{-1})\left[(T^{-1})^{*m}(T^{-1})^n(T^{-1})^n(T^{-1})^{*m}\right] = \left[(T^{-1})^n(T^{-1})^{*m}(T^{-1})^{*m}(T^{-1})^n\right](T^{-1})$$

Therefore T^{-1} is also skew (n,m) binormal □

Proposition 2.2. *Every skew (n,m) normal operator $T \in B(H)$ is skew (n,m) binormal*

Proof. Let $T \in B(H)$ be skew (n,m) normal operator, then by definition

$$(T^{*m}T^n)T = T(T^nT^{*m})$$

We show that T is skew (n,m) binormal.

$$\begin{aligned}(T^{*m}T^nT^nT^{*m})T &= T^{*m}T^nT^nT^{*m}T\\ &= T^{*m}T^n(T^nT^{*m}T)\\ &= T^{*m}T^n(TT^{*m}T^n) && \text{by skew } (n,m) \text{ normality of } T\\ &= (T^{*m}T^nT)T^{*m}T^n\\ &= T(T^nT^{*m}T^{*m}T^n)\end{aligned}$$

Hence T is skew (n,m) binormal. □

Theorem 2.3. *Let T be skew (n,m) normal operator, then $T \in B(H)$ is skew $(n+k(n-1),m)$ normal for every $k \in \mathbb{N}$.*

Proof. We give the proof by induction.
Base case $k = 1$.

$$\begin{aligned}(T^{n+(n-1)}T^{*m})T &= T^{n-1}(T^nT^{*m})T\\ &= T^{n-1}T(T^{*m}T^n)\\ &= (T^nT^{*m})TT^{n-1} && \text{by skew } (n,m) \text{ normality of } T\\ &= T(T^{*m}T^n)T^{n-1}\\ &= T(T^{*m}T^{n+(n-1)})\end{aligned}$$

Inductive step: Assume the result holds for $n = k$.

To prove the result for $n = k+1$

$$\begin{aligned}(T^{n+(k+1)(n-1)}T^{*m})T &= (T^{n+k(n-1)+(n-1)}T^{*m})T\\ &= T^{n-1}(T^{n+k(n-1)}T^{*m})T\\ &= T^{n-1}T(T^{*m}T^{n+k(n-1)})\\ &= (T^nT^{*m})TT^{n+k(n-1)-1} && \text{by skew } (n,m) \text{ normality of } T\\ &= T(T^{*m}T^n)T^{n+k(n-1)-1}\\ &= T(T^{*m}T^n)T^{(k+1)(n-1)}\\ &= T(T^{*m}T^{n+(k+1)(n-1)})\end{aligned}$$

Therefore T is skew $(n+k(n-1),m)$ normal. □

Remark. *The product of two skew (n, m) normal operator is not skew (n, m) normal operator in general.*

Consider the following example

Example 2.4. *Let* $T = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ *and* $S = \begin{bmatrix} i & i & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ *it can easily be shown that both T and S are skew $(1,1)$ normal operators. We now check their product,*

$$(TS) = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i & i & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

checking if (TS) is skew $(1,1)$ normal, we have

$$(TS)\left[(TS)^*(TS)\right] = \begin{bmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} -2 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \left[(TS)(TS)^*\right](TS)$$

Hence, the product of two skew (n, m) normal operators is not necessarily skew (n, m) normal.

Theorem 2.5. *Let $T \in B(H)$ be a normal operator and S be skew (n, m) normal operator. If T and S commute, then TS is skew (n, m) normal operator.*

Proof. Consider

$$\left[(TS)^n(TS)^{*m}\right](TS) = (T^nS^nS^{*m}T^{*m})TS$$

Since T is a normal operator which commutes with S, then by the Fuglede-Putnam theorem, S commutes with T^*, therefore

$$\begin{aligned}
\left[(TS)^n(TS)^{*m}\right](TS) &= \left[T^nS^nS^{*m}T^{*m}\right](TS) \\
&= T^n(S^nS^{*m})T^{*m}ST \\
&= T^n(S^nS^{*m})ST^{*m}T \qquad \text{by the Fuglede-Putnam theorem} \\
&= T^nSS^{*m}S^nT^{*m}T \\
&= ST^nS^{*m}S^nT^{*m}T \\
&= SS^{*m}T^nS^nT^{*m}T \\
&= SS^{*m}S^nT^nT^{*m}T \\
&= SS^{*m}S^nTT^{*m}T^n \\
&= SS^{*m}TS^nT^{*m}T^n \\
&= STS^{*m}S^nT^{*m}T^n \\
&= STS^{*m}T^{*m}S^nT^n \\
&= TS\left[(TS)^{*m}(TS)^n\right]
\end{aligned}$$

□

Definition 2.1. An operator $T \in B(H)$ is quasi (n, m) normal, if $T(T^{*m}T^n) = (T^{*m}T^n)T$ and k-quasi (n, m) normal if $T^k(T^{*m}T^n) = (T^{*m}T^n)T^k$

Theorem 2.6. *Let T be (n, m) normal operator and quasi (n, m) normal operator, then T is skew (n, m) normal*

Proof. Let T be (n, m) normal operator, therefore

$$T^n T^{*m} = T^{*m} T^n$$

Since T is quasi (n, m) normal, we have

$$T(T^{*m} T^n) = (T^{*m} T^n)T$$

Then

$$T(T^{*m} T^n) = (T^{*m} T^n)T = (T^n T^{*m})T$$

□

The converse of the theorem is however not true. Consider the following example

Example 2.7. *Let* $T = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ *then* $T^* = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ *Simple computation shows that*

$$TT^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = T^*T.$$

Hence T is not $(1, 1)$ normal. For quasi $(1, 1)$ normal, observe that

$$T(T^*T) = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = (T^*T)T.$$

Therefore T is not quasi $(1, 1)$ normal. We now check whether T is skew (n, m) normal, we require that $(TT^*)T = T(T^*T)$. We have

$$(TT^*)T = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = T(T^*T)$$

Therefore, T is skew (n, m) normal.

Theorem 2.8. *Let T be k-skew (n, m) normal operator which is unitarily equivalent to S. Then S is also k-skew (n, m) normal operator*

Proof. Since T is unitarily equivalent to S, then there exists a unitary operator U such that $S = U^*TU$ and $S^* = (U^*TU)^* = U^*T^*U$.

$$\begin{aligned}
(S^n S^{*m})S^k &= U^*T^nUU^*T^{*m}UU^*T^kU \\
&= U^*T^nT^{*m}T^kU \\
&= U^*T^k(T^{*m}T^n)U \qquad \text{since } T \text{ is } k\text{-skew } (n, m) \text{ normal operator} \\
&= U^*T^kUU^*T^{*m}UU^*T^nU \\
&= S^k(S^{*m}S^n)
\end{aligned}$$

Hence S is k-skew (n, m) normal operator □

Lemma 2.9. *Let T be skew (n, m) normal operator which is unitarily equivalent to S. Then S is also skew (n, m) normal operator*

Proof. The proof is trivial, hence ommited. □

Theorem 2.10. *Let T be k-quasi (n, m) normal operator which is unitarily equivalent to S. Then S is also k-quasi (n, m) normal operator*

Proof. Recall that an operator T is k-quasi (n, m) normal, if

$$T^k(T^{*m}T^n) = (T^{*m}T^n)T^k \qquad \text{for} \quad k, m, n \in \mathbb{Z}$$

Therefore

$$\begin{aligned} S^k(S^{*m}S^n) &= U^*T^kU(U^*T^{*m}UU^*T^nU) \\ &= U^*T^k(T^{*m}T^n)U \\ &= U^*(T^{*m}T^n)T^kU \qquad \text{since } T \text{ is } k\text{-quasi } (n,m) \text{ normal} \\ &= U^*T^{*m}UU^*T^nUU^*T^kU \\ &= (S^{*m}S^n)S^k \end{aligned}$$

Hence S is k-quasi (n, m) normal □

Theorem 2.11. *Let T be k-quasi (n, m) binormal operator which is unitarily equivalent to S. Then S is also k-quasi (n, m) binormal operator*

Proof. An operator T is k-quasi (n, m) binormal, if

$$T^k(T^{*m}T^nT^nT^{*m}) = (T^{*m}T^nT^nT^{*m})T^k \qquad \text{for} \quad k, m, n \in \mathbb{Z}$$

Therefore

$$\begin{aligned} S^k(S^{*m}S^nS^nS^{*m}) &= U^*T^kU(U^*T^{*m}UU^*T^nUU^*T^nUU^*T^{*m}U) \\ &= U^*T^k(T^{*m}T^nT^nT^{*m})U \\ &= U^*(T^{*m}T^nT^nT^{*m})T^kU \qquad \text{since } T \text{ is } k\text{-quasi } (n,m) \text{ binormal} \\ &= U^*T^{*m}UU^*T^nUU^*T^nUU^*T^{*m}UU^*T^kU \\ &= (S^{*m}S^nS^nS^{*m})S^k \end{aligned}$$

Hence S is k-quasi (n, m) binormal □

Theorem 2.12. *Let S and T be k-skew (n, m) binormal and doubly commuting operators. Then ST is k-skew (n, m) binormal operator.*

Proof. Since S and T are k-skew (n, m) binormal, we have

$$(S^{*m}S^nS^nS^{*m})S^k = S^k(S^nS^{*m}S^{*m}S^n)$$

and

$$(T^{*m}T^nT^nT^{*m})T^k = T^k(T^nT^{*m}T^{*m}T^n)$$

In addition, the operators are doubly commuting, this implies

$$ST = TS \quad \text{and} \quad ST^* = T^*S$$

taking the adjoint on both sides yields

$$S^*T^* = T^*S^* \quad \text{and} \quad TS^* = S^*T$$

respectively. We now have;

$$\begin{aligned}
\left[(ST)^{*m}(ST)^{n}(ST)^{n}(ST)^{*m}\right](ST)^{k} &= S^{*m}T^{*m}S^{n}T^{n}S^{n}T^{n}S^{*m}T^{*m}S^{k}T^{k} \\
&= S^{*m}T^{*m}S^{n}T^{n}S^{n}T^{n}S^{*m}S^{k}T^{*m}T^{k} \\
&= S^{*m}T^{*m}S^{n}T^{n}S^{n}S^{*m}T^{n}S^{k}T^{*m}T^{k} \\
&= S^{*m}S^{n}T^{*m}S^{n}T^{n}S^{*m}S^{k}T^{n}T^{*m}T^{k} \\
&= S^{*m}S^{n}S^{n}T^{*m}S^{*m}T^{n}S^{k}T^{n}T^{*m}T^{k} \\
&= S^{*m}S^{n}S^{n}S^{*m}T^{*m}S^{k}T^{n}T^{n}T^{*m}T^{k} \\
&= \left[(S^{*m}S^{n}S^{n}S^{*m})S^{k}\right]\left[(T^{*m}T^{n}T^{n}T^{*m})T^{k}\right] \\
&= \left[S^{k}(S^{n}S^{*m}S^{*m}S^{n})\right]\left[T^{k}(T^{n}T^{*m}T^{*m}T^{n})\right] \\
&= S^{k}S^{n}S^{*m}S^{*m}T^{k}S^{n}T^{n}T^{*m}T^{*m}T^{n} \\
&= S^{k}S^{n}S^{*m}T^{k}S^{*m}T^{n}S^{n}T^{*m}T^{*m}T^{n} \\
&= S^{k}S^{n}T^{k}S^{*m}T^{n}S^{*m}T^{*m}S^{n}T^{*m}T^{n} \\
&= S^{k}T^{k}S^{n}T^{n}S^{*m}T^{*m}S^{*m}T^{*m}S^{n}T^{n} \\
&= (ST)^{k}\left[(ST)^{n}(ST)^{*m}(ST)^{*m}(ST)^{n})\right]
\end{aligned}$$

We therefore conclude that ST is k-skew (n, m) binormal □

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Competing Interests

Authors have declared that no competing interests exist.

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