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A General Inequality Related to Variational Inequalities and Its Consequences

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Abstract

Let X be a Hausdorff topological vector space with dual X^*, K a nonempty subset of X, and $f: K\times K \to \mathbb{R}$ be any map. In this paper we study the following problem: "*Find* $x_0 \in K$ *such that* $f(x_0, y) \geq 0$ *for all* $y \in K$". Some results on this problem have been studied by Takahasi, Park and Kim and Yen. Behera and Panda proved several results on this problem in the setting of a Banach space. This problem includes as special cases, many problems on variational inequalities and generalized variational inequalities studied by many authors. As a consequence of the main results, we also consider the problem, "*Find* $x_0 \in K$ *such that* $f(x_0, y) \geq 0$ *for all* $y \in S(x_0)$" where $S: K \to 2^K$ is any point-to-set map which, includes as a special case, the classical quasi-variational inequality problem. A generalization of Minty's Lemma is also studied.

Keywords: Partition of unity, upper and lower semi continuous point-to-set map, monotone operator.

1 Introduction

Let X be a Hausdorff topological vector space with dual X^* and K a nonempty closed and convexsubset of X. Let the value of $u \in X^*$ at $x \in X$ be denoted by (u, x). Let $g: K \to \mathbb{R}$ be a map (possibly nonlinear). The classical minimization problem for the pair (g, K) is to find $x_0 \in K$ such that

$$g(x_0) = \min_{y\in K} g(y).$$

If we define a function $f: K \times K \to \mathbb{R}$ as $f(x, y) = g(y) - g(x)$ for all $x, y \in K$, then the above problem reduces to the problem of finding $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in K$.

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If $T: K \to X^*$, then the nonlinear variational inequality problem is to find $x_0 \in K$ such that $(Tx_0, y - x_0) \geq 0$ for all $y \in K$. If we define $f: K \times K \to \mathbb{R}$ as $f(x, y) = (Tx, y - x)$, then it reduces to the problem of finding $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in K$.

Motivated by the above facts, Behera and Panda [3] considered the following problem:

P_1: *Given* $f: K \times K \to \mathbb{R}$, *find* $x_0 \in K$ *such that* $f(x_0, y) \geq 0$ *for all* $y \in K$.

Certain variation of this problem is known as equilibrium problem. The readers are advised to refer [5], [6] and [13] for a detailed discussion on equilibrium problem and its generalization. Some similar results are also available in [20].

1.1 Special Cases

1. If $z \in K$ is fixed and $f(x, y) = (T((z + x)/2), y - x)$, then P_1 reduces to the problem of finding $x_0 \in K$ such that $(T((z + x_0)/2), y - x_0) \geq 0$ for all $y \in K$, which is the *variational-type inequality problem* originally introduced by Behera and Panda [2].

2. If $f(x, y) = (Tx - Ax, y - x)$ for some maps $T, A : K \to X^*$, then P_1 reduces to the problem of finding $x_0 \in K$ such that $(Tx_0, y - x_0) \geq (Ax_0, y - x_0)$ for all $y \in K$ which is the *strongly nonlinear variational inequality problem* studied by Nanda[10] and Noor[11].

3. If $f(x, y) = (Tx - Ax, g(y) - g(x))$ for some maps $T, A: K \to X^*$, $g: K \to K$, then P_1 reduces to the problem of finding $x_0 \in K$ such that $(Tx_0, g(y) - g(x_0)) \geq (Ax_0, g(y) - g(x_0))$ for all $y \in K$, which is the *strongly nonlinear implicit variational inequality problem* introduced and studied by Noor[12] in connection with the solution of the differential equations of odd order.

Some results on Problem P_1 are available in Takahasi [18] and Park and Kim [14]. Behera and Panda [3] proved several results on the existence of solution of this problem in the setting of Banach spaces. Many authors used results on Problem P_1 to prove the existence of solutions of variational and generalized variational inequality problems.

In this paper, we prove some results on the existence of solution of Problem P_1 under different assumptions in the setting of Hausdorff topological vector spaces. We also use a result on the existence of solution of Problem P_1 for the study the following problem, which is a generalization of the quasi-variational inequality problem introduced by Benoussan and Lions [4] in connection with impulse control and subsequently studied by Baicchi and Capello [1] and Mosco [9].

> **P_2**: *Find* $x_0 \in K$ *such that* $f(x_0, y) \geq 0$ *for all* $y \in S(x_0)$ *where* $S: K \to 2^K$ *is any point-to-set map*.

We also prove a theorem on the uniqueness of solution of Problem P_1 which would serve as a generalization of the well-known Minty's Lemma (see [7], p.6).

2 Priliminaries

In this section we recall some definitions and known results which will be needed in the sequel. Throughout this section, X is a real Hausdorff topological vector space with dual X^* and K a nonempty convex subset of X.

2.1 Definition

A mapping $T: K \to X^*$ is said to be monotone if $(Tx - Ty, x - y) \geq 0$ for all $x, y \in K$. T is said to be strictly monotone if strict inequality holds whenever $x \neq y$. T is said to be pseudo-monotone if $x, y \in K$ and $(Tx, x - y) \geq 0$ then $(Ty, x - y) \geq 0$.

2.2 Definition

A function $h: K \to \mathbb{R}$ is said to be upper semi-continuous, if for each real number λ, the set $\{x \in K: h(x) < \lambda\}$ is open; f is said to be lower semi-continuous if $-f$ is upper semi-continuous.

2.3 Definition

A function $h: K \to \mathbb{R}$ is said to be quasi-convex if for each real number λ, the set $\{x \in K: h(x) < \lambda\}$ is convex; function f is said to be quasi-concave if $-f$ is quasi-convex.

2.4 Definition

Let M and N be two topological spaces, and $h: M \to 2^N$, a point-to-set map. Then h is said to be upper semi-continuous at $x_0 \in M$ if for each open set U of N with $h(x_0) \subset U$, there exists a neighborhood $n(x_0)$ of x_0 in M such thatif $x \in n(x_0)$ then $h(x) \subset U$; h is said to be lower semi-continuous at $x_0 \in M$ if for each open set U of N with $h(x_0) \cap U \neq \emptyset$, there exists a neighborhood $n(x_0)$ of x_0 in M such that if $x \in n(x_0)$ then $h(x) \cap U \neq \emptyset$; h is said to be upper semi-continuous (lower semi-continuous) on M *if* h is upper semi-continuous (lower semi-continuous) at each point of M.

2.5 Theorem

(Isac [8]) Let K be a nonempty compact convex subset of a Hausdorff topological vector space. Let f and g be two real valued functions on $K \times K$ having the following properties:

(a) *For all* $x, y \in K$, $f(x, y) \leq g(x, y)$ and $g(x, x) \leq 0$,
(b) *For each fixed* $y \in K$, $f(x, y)$ *is a lower semicontinuous function of* x *on* K,
(c) *For each fixed* $x \in K$, $g(x, y)$ *is a quasi-concave function of* x *on* K.

Then there exists a point $\hat{x} \in K$ *such that,* $f(\hat{x}, y) \leq 0$ *for all* $y \in K$.

2.6 Theorem

(Shih and Tan [17]). *Let* X*be a Hausdorff topological vector space with dual* X^**and* K *a nonempty convex subset of* X. *Let* $S: K \to 2^K$ *be an upper semicontinuous point-to-set map such that for each* $x \in K$, $S(x)$ *is nonempty and bounded. Then for each* $p \in X^*$, *the map* $f_p: K \to \mathbb{R}$ *defined by* $f_p(x) = \sup_{y \in S(x)}(p, y)$ is *upper semicontinuous.*

2.7 Theorem

(Takahasi [18])*Let* K *be a nonempty compact convex subset of a locally convex Hausdorff topological vector space* X *and let* $F: K \times K \to \mathbb{R}$ *be a function such that*

(a) *for each* $y \in K$, *the map* $x \mapsto F(x, y)$ *is upper semicontinuous,*
(b) *for each* $x \in K$, *the map* $y \mapsto F(x, y)$ *is convex,*
(c) $F(x, x) \geq \alpha$ *for all* $x \in K$ *with some real number* α.

Then there exists$x_0 \in K$ *such that* $F(x_0, y) \geq \alpha$ *for all* $y \in K$.

2.8 Theorem

(Rudin [15]) *Let* K *be a compact subset of a topological space and let* $\{V_1, V_2, \cdots, V_n\}$ *be a finite open covering of* K. *Then there exists a family* $\{\beta_1, \beta_2, \cdots, \beta_n\}$ *of continuous real valued functions on* K *such that* $\beta_i(x) = 0$ *outside* V_i, $0 \leq \beta_i(x) \leq 1$ *for all* $i \in \{1, 2, \cdots, n\}$ *and for all* $x \in K$, *and* $\sum_{i=1}^{n} \beta_i(x) = 1$ *for all* $x \in K$.

2.9 Theorem

(Tarafdar [19]) *Let* K *be a nonempty subset of a Hausdorff topological vector space* X *and let* $F: K \to 2^K$ *be a point-to-set map such that*

(a) *for each* $x \in K$,$F(x)$ *is nonempty,*
(b) *for each* $y \in K$, $F^{-1}(y) = \{x \in K: y \in F(x)\}$ *contains a relatively open subset* A_y *of* K(A_y *may be empty for some* y),
(c) $\bigcup_{x \in K} A_x = K$,
(d) K *contains a nonempty set* K_0 *contained in a compact convex subset* K_1 *of* K *such that* $D = \bigcap_{x \in K_0} A_x^C$ *is compact* (D *may be empty*).

Then there exists $x_0 \in K$ such that $x_0 \in F(x_0)$.

2.10 Theorem

([17], p. 34). *Let* X *be a Hausdorff topological vector space with dual* X^* *and* K *a nonempty convex subset of* X. *Let* $S: K \to 2^K$ *be a point-to-set upper semi-continuous map such that for each* $x \in K$, $S(x)$ *is nonempty and bounded. Then for each* $p \in X^*$ *the map* $f_p: K \to \mathbb{R}$ *defined by* $f_p(x) = \sup_{y \in S(x)} (p, y)$ *is upper semi-continuous.*

2.11 Theorem

([16], p.201). *Let X be a locally convex Hausdorff topological vector space with dual X^* and K a nonempty closed and convex subset of X. Let x^*be a point of X not in K. Then there exists $p \in X^*$ such that* $(p, x^*) > \sup_{x \in K}(p, x)$.

3 Existence of Solution

The following theorems on the existence of solution of problem P_1 are the main results of this paper.

3.1 Theorem

Let K be a nonempty compact convex subset of a Hausdorff topological vector space X and let $f: K \times K \to \mathbb{R}$ be a map such that

(a) $f(x, x) \geq 0$*for each $x \in K$,*
(b) *the map $y \longmapsto f(x, y)$of K into $\mathbb{R}$ is quasi-convex for each $x \in K$,*
(c) *for each $y \in K$, the set $\{x \in K : f(x, y) < 0\}$ is open in K.*

Then there exists $x_0 \in K$ such that$f(x_0, y) \geq 0$ for all $y \in K$.

*Proof.*Same as the proof of Theorem 2.7. Indeed Takahasi [18] assumed that for each $y \in K$, the map $x \longmapsto f(x, y)$ of K into $\mathbb{R}$ is upper semi-continuous and for each $x \in K$, the map $y \longmapsto f(x, y)$ of K into $\mathbb{R}$ is convex which are stronger then hypotheses (b) and (c) of this theorem.

The following theorem generalizes Theorem 3.1 to an arbitrary nonempty convex subset K of X.

3.2 Theorem

Let K be a nonempty convex subset of a Hausdorff topological vector space X and let $f: K \times K \to \mathbb{R}$ be a map such that

(a) $f(x, x) \geq 0$ *for each $x \in K$,*
(b) *the map $y \longmapsto f(x, y)$ of K into $\mathbb{R}$ is quasi-convex for each $x \in K$,*
(c) *for each $y \in K$, the set $\{x \in C: f(x, y) < 0\}$ is open in K for every compact subset C of K,*
(d) *there exists a nonempty compact subset L of K such that for each $x \in K - L$, there exists $u \in L$ for which $f(x, u) < 0$.*

Then there exists $x_0 \in L$ such that $f(x_0, y) \geq 0$ for all $y \in K$.

*Proof.*For each $y \in K$, let
$$A(y) = \{x \in L : f\ (x, y)\ \geq\ 0\}.$$

From hypothesis (c) it follows that $A(y)$ is a nonempty compact subset of K. The conclusion of the theorem follows if $\bigcap_{y\in K} A(y) \neq \emptyset$. To ascertain this, it is sufficient to show that the family $\{A(y) : y \in K\}$ has the finite intersection property.

Let $y_1, y_2, \cdots, y_n$ be arbitrary elements of $K$and K_0, the convex hull of $L \cup \{y_1, y_2, \cdots, y_n\}$. Then K_0 is a nonempty compact convex subset of K. By Theorem 3.1, there exists $\tilde{x}_0 \in K_0$ such that

$$f(\tilde{x}_0, y) \geq 0 \qquad (1)$$

for all $y \in K_0$. We claim that $\tilde{x}_0 \in L$; for if $\tilde{x}_0 \notin L$, then, $\tilde{x}_0 \in K_0 - L \subset K - L$ and by hypothesis (d), there exists $u \in L$ such that $f(\tilde{x}_0, u) < 0$ which contradicts (1) when $y = u$. Thus $\tilde{x}_0 \in L$, and in particular, $\tilde{x}_0 \in A(y_i)$ for i=$1, 2, \cdots, n$; that is $\tilde{x}_0 \in \bigcap_{i=1}^{n} A(y_i)$. Hence, $\bigcap_{i=1}^{n} A(y_i) \neq \emptyset$ which proves that the family $\{A(y) : y \in K\}$ has the finite intersection property. This completes the proof.

If X is a normed linear space and K is locally compact, we have the following result.

3.3 Theorem

Let K be a nonempty locally compact convex subset of a real normed linear space X with $0 \in K$, *and let $f: K \times K \to \mathbb{R}$ be a map such that*

(a) *the map $y \mapsto f(x, y)$of K into $\mathbb{R}$ is quasi-convex for each $x \in K$,*
(b) *for all$y \in K$ and $r > 0$, the set $\{x \in K_r : f(x, y) < 0\}$is open in K, where $K_r = \{x \in K: \|x\| \leq r\}$,*
(c) *$f(x, x) = 0$ for each $x \in K_r$ and for each $r > 0$,*
(d) *there exists $r > 0$ such that $f(x, 0) < 0$ whenever $\|x\| = r$.*

Then there exists $x_0 \in K$such that$f(x_0, y) \geq 0$ for all $y \in K$.

Proof. For each $r > 0$, the set K_r is a nonempty compact and convex subset of K. Hence by Theorem 3.1 there exists $x_r \in K_r$ such that

$$f(x_r, y) \geq 0 \qquad (2)$$

for all $y \in K_r$. We claim that $\|x_r\| < r$; for if $\|x_r\| = r$, then by hypothesis (d), $f(x_r, 0) < 0$ which contradicts (2) when $y = 0$. Now let $x \in K$ be arbitrary and choose $\lambda > 0$ sufficiently small so that

$$y_r = \lambda x + (1\text{–}\lambda) x_r \in K_r.$$

Now by hypotheses (a) and (b)

$$0 \leq f(x_r, y_r) = f(x_r, \lambda x + (1\text{–}\lambda) x_r)$$

$$\leq \lambda f(x_r, x) + (1\text{–}\lambda) f(x_r, x_r) = \lambda f(x_r, x).$$

Since $\lambda > 0$, it follows that $f(x_r, x) \geq 0$. Since x is arbitrary, the proof is complete.

If X is a reflexive Banach space then local compactness of K can be relaxed from the above theorem.

3.4 Theorem

Let K be a nonempty closed and convex subset of a reflexive real Banach space X with $0 \in K$ and $f: K \times K \to \mathbb{R}$ be a map such that

(a) $f(x,x) = 0$ *for each* $x \in K$,
(b) *the map* $y \longmapsto f(x,y)$ *of* K *into* $\mathbb{R}$ *is convex for each* $x \in K$,
(c) *the set* $\{x \in C : f(x,y) < 0\}$ *is weakly open in* K *for each* $y \in K$, *and for each weakly compact subset* C *of* K.

Then there exists $x_0 \in K$ *such that* $f(x_0, y) \geq 0$ *for all* $y \in K$ *under each of the following conditions*:
(i) K *is bounded*,
(ii) *there exists* $r > 0$ *such that* $f(x,0) < 0$ *whenever* $\|x\| = r$.

Proof. Equip K with the weak topology. If K is bounded, then it becomes weakly compact and in this case, the proof follows from Theorem 3.1.

If K is not bounded let

$$K_r = \{x \in K: \|x\| \leq r\}.$$

Then K_r is a nonempty closed, convex and bounded subset of K. By the first part of the theorem, there exists $x_r \in K_r$ such that $f(x_r, y) \geq 0$ for all $y \in K_r$. The remaining part of the proof follows from the proof of Theorem 3.3 and hence it is omitted.

The following theorem generalizes Theorem 3.4 to a Hausdorff topological vector space.

3.5 Theorem

Let K be a nonempty convex subset of a Hausdorff topological vector space X and L, a nonempty compact and convex subset of K. Let $f: K \times K \to \mathbb{R}$ be a map such that

(a) $f(x,x) = 0$ *for each* $x \in L$,
(b) *the map* $y \longmapsto f(x,y)$ *of* K *into* $\mathbb{R}$ *is convex for each* $x \in K$,
(c) *for each* $y \in K$, *the set* $\{x \in L : f(x,y) < 0\}$ *is open in* L,
(d) *for every* x *in the boundary of* L, *there exists* $u \in L$ *such that* $f(x,u) < 0$.

Then there exists $x_0 \in L$ *such that* $f(x_0, y) \geq 0$ *for all* $y \in K$.

Proof. By theorem 3.1 there exists $x_0 \in L$ such that

$$f(x_0, y) \geq 0 \tag{3}$$

for all $y \in L$. We claim that $x_0 \in \text{int}L$ (where $\text{int}L$ stands for the interior of L), for if $x_0 \notin \text{int}L$, then by hypothesis (d) there exists $u \in L$ such that $f(x_0, u) < 0$ which contradicts (3) when $y = u$.

Now let $x \in K$ be arbitrary and choose $\lambda > 0$ sufficiently small so that

$$y_\lambda = \lambda x + (1-\lambda)x_0 \in L.$$

By hypotheses (a) and (b)

$$0 \leq f(x_0, y_\lambda) \leq \lambda f(x_0, x) + (1-\lambda) f(x_0, x_0) = \lambda f(x_0, x).$$

Since $\lambda > 0$, it follows that $f(x_0, x) \geq 0$. This completes the proof.

The following theorem is a variant of Theorem 3.1 in which, the hypotheses (a) and (b) of Theorem 3.1 have been replaced by a single generalized condition.

3.6 Theorem

Let K be a nonempty compact convex subset of a Hausdorff topological vector space X and let $f: K \times K \to \mathbb{R}$ *be a map such that*

(a) *for each* $y \in K$, *the set* $\{x \in K : f(x, y) < 0\}$ *is open in* K,
(b) *for any finite* subset $\{y_1, y_2, \cdots, y_n\}$ *of* K *and* $u \in \text{conv}\,(\{y_1, y_2, \cdots, y_n\}), \max_{1 \leq i \leq n} f(u, y_i) \geq 0$.

Then there exists $x_0 \in K$ *such that* $f(x_0, y) \geq 0$ for all $y \in K$.

Proof. Assume that the assertion is false. Then for each $x \in K$, there exists $y \in K$ such that $f(x, y) < 0$. For each $y \in K$, let

$$A(y) = \{x \in K : f(x, y) < 0\}.$$

Then by hypothesis (a), $A(y)$ is open for every $y \in K$. Further $K = \bigcup_{y \in K} A(y)$. Since K is compact, there exists $\{y_1, y_2, \cdots, y_n\} \subset K$ such that $K = \bigcup_{i=1}^{n} A(y_i)$. Let $\{\beta_1, \beta_2, \cdots, \beta_n\}$ be a partition of unity subordinate to the covering $\{A(y_i) : i = 1, 2, \cdots, n\}$ and let $S = \text{conv}(\{y_1, y_2, \cdots, y_n\})$. Then S is a compact convex subset of K. Define a continuous map $p: K \to K$ as $p(x) = \sum_{i=1}^{n} \beta_i(x) y_i$. Since $p(x)$ is a convex linear combination of points of the set S, $p(x) \in S$ for each $x \in K$. In particular, p maps S into S. By Browuer's fixed point theorem p has a fixed point; that is, there exists $\hat{x} \in K$ such that $\hat{x} = p(\hat{x})$.

Since $\{\beta_1, \beta_2, \cdots, \beta_n\}$ is a partition of unity, $\beta_i(\hat{x}) > 0$ for at least one i. But $\beta_i(\hat{x}) > 0$ implies that $\hat{x} \in A(y_i)$ so that

$$f(\hat{x}, y_i) < 0. \tag{4}$$

If

$$J = \{i: 1 \leq i \leq n \text{and} \beta_i(\hat{x}) > 0\}$$

then

$$\hat{x} = p(\hat{x}) = \sum_{i=1}^{n} \beta_i(\hat{x}) y_i = \sum_{i \in J} \beta_i(\hat{x}) y_i$$

so that $\hat{x} \in \text{conv}\{y_i: i \in J\}$. Now by hypothesis (b), there exists $i_0 \in J$ such that $f(\hat{x}, y_{i_0}) \geq 0$ which contradicts (4). This contradiction proves the theorem.

The following theorem generalizes Theorem 3.6 to an arbitrary nonempty convex subset K of X.

3.7 Theorem

Let X be a locally convex Hausdorff topological vector space, K a nonempty convex subset of X, K_0 a nonempty compact convex subset of K, and L a nonempty compact subset of K. Let $f: K \times K \to \mathbb{R}$ be a map such that

(a) *for each $y \in K$, the set $\{x \in L: f(x, y) \geq 0\}$ is closed in K,*
(b) *for any finite subset $\{y_1, y_2, \cdots, y_n\}$ of K, and* $u \in \text{conv}(\{y_1, y_2, \cdots, y_n\}), \max_{1 \leq i \leq n} f(u, y_i) \geq 0$.
(c) *for each $x \in K - L$, there exists $u \in \text{conv}(K_0 \cup \{x\})$ such that $f(x, u) < 0$.*

Then there exists $x_0 \in L$ such that $f(x_0, y) \geq 0$ for all $y \in K$.

Proof. For each $y \in K$, let

$$A(y) = \{x \in L: f(x, y) \geq 0\}.$$

From (b) it follows that $A(y)$ is closed and consequently compact for each $y \in K$. The assertion of the theorem follows if $\bigcap_{y \in K} A(y) \neq \emptyset$. For this, it is sufficient to prove that the family $\{A(y): y \in K\}$ has the finite intersection property.

Let $\{y_1, y_2, \cdots, y_n\} \subset K$ and $S = \text{conv}(K_0 \cup \{y_1, y_2, \cdots, y_n\})$. Clearly S is a non-empty compact convex subset of K. By Theorem 3.6 there exists $\hat{x} \in S$ such that

$$f(\hat{x}, y) \geq 0 \tag{5}$$

for all $y \in S$. We claim that $\hat{x} \in L$; for if $\hat{x} \notin L$, that is, $\hat{x} \in S - L \subset K - L$, then there exists $u \in \text{conv}(K_0 \cup \{\hat{x}\})$ such that $f(\hat{x}, y) < 0$ which contradicts (5) when $y = u$. Thus $\hat{x} \in L$, and in particular, $\hat{x} \in A(y_i)$ for each $i \in \{1, 2, \cdots, n\}$, that is, $\tilde{x}_0 \in \bigcap_{i=1}^{n} A(y_i)$. Hence, $\bigcap_{i=1}^{n} A(y_i) \neq$

$\emptyset$showing that the family $\{A(y) : y \in K\}$ has the finite intersection property. This completes the proof.

The following theorem also deals with the existence of solution of Problem P_1 under pseudo-monotonicity type assumptions.

3.8 Theorem

Let K be a nonempty closed and convex subset of a Hausdorff topological vector space X and let $f: K \times K \to \mathbb{R}$ be a map such that

- *(a)* $f(x,x) \geq 0$ *for each* $x \in K$,
- *(b) the map* $y \mapsto f(x,y)$ *of K into $\mathbb{R}$ is convex and lower semicontinuous for each* $x \in K$,
- *(c) the map* $x \mapsto f(x,y)$ *of K into $\mathbb{R}$ is lower semicontinuous from the line segments of K to the usual topology of $\mathbb{R}$,*
- *(d)* $x \in K, y \in K$ *and* $f(x,y) \geq 0$ *implies* $f(y,x) \leq 0$,
- *(e) there exists a nonempty set K_0 contained in a compact convex subset K_1 of K such that* $D = \bigcap_{y \in K_0} \{y \in K: f(x,y) \leq 0\}$ *is either compact or empty.*

Then there exists $x_0 \in K$*such that*

$$f(x_0, y) \geq 0 \tag{6}$$

For all $y \in K$.

Note: If we set $f(x,y) = (Tx, y - x)$ for some operator $T: K \to X^*$ then condition (d) of the above theorem is equivalent to the fact that T is pseudo-monotone.

To prove the above theorem, we need the following lemma.

3.9 Lemma

Let K be a nonempty closed and convex subset of a Hausdorff topological vector space X and let $f: K \times K \to \mathbb{R}$ be a map such that

- (a) $f(x,x) \geq 0$ *for each* $x \in K$,
- (b) *the map* $y \mapsto f(x,y)$ *of K into $\mathbb{R}$ is convex for each* $x \in K$,
- (c) *the map*$x \mapsto f(x,y)$ *of K into$\mathbb{R}$ is lower semicontinuous from the line segments of K to the usual topology of $\mathbb{R}$,*
- (d) $x \in K$, $y \in K$ *and* $f(x,y) \geq 0$ *implies* $f(y,x) \leq 0$.

Then the following are equivalent:

- *(A)* $x_0 \in K$*and* $f(x_0, y) \geq 0$ *for all* $y \in K$.
- *(B)* $x_0 \in K$*and* $f(y, x_0) \leq 0$ *for all* $y \in K$.

Note: If we set $f(x,y) = (Tx, y - x)$ for some operator $T: K \to X^*$, then Lemma 3.9 reduces to Minty's Lemma ([7], p. 6).

Proof. Assume that (A) holds. Then for some $x_0 \in K$ and for all $y \in K$, $f(x_0, y) \geq 0$. By hypothesis (d), $f(y, x_0) \leq 0$ for all $y \in K$ from which (B) follows.

Now assume that (B) holds. Then for some $x_0 \in K$, $f(y, x_0) \leq 0$ for all $y \in K$. Fix $y \in K$, $t \in (0,1)$ and set $y_t = (1-t)x_0 + ty$. Then $y_t \in K$ and by (B), $f(y_t, x_0) \leq 0$ for each $t \in (0,1)$. Now it follows from hypotheses (a) and (b) that

$$0 \leq f(y_t, y_t) \leq (1-t)f(y_t, x_0) + tf(y_t, y).$$

Thus

$$f(y_t, y) \geq -\frac{1-t}{t} f(y_t, x_0) \geq 0$$

for each $t \in (0,1)$. Since the map $x \mapsto f(x,y)$ of K into $\mathbb{R}$ is lower semicontinuous from the line segments of K to the usual topology of $\mathbb{R}$, taking limit $t \to 0^+$ in the above inequality, we get $f(x_0, y) \geq 0$. Since $y \in K$ is arbitrary (B) follows. This completes the proof of the lemma.

3.10 Proof of Theorem 3.8

Assume that inequality (6) has no solution. Then, for each $x \in K$, there exists $u \in K$ such that

$$f(x,u) < 0 \tag{7}$$

and by Lemma 3.9, there exists $v \in K$ such that

$$f(v,x) > 0. \tag{8}$$

Thus for each $x \in K$, it follows from (7) that the set

$$F(x) = \{y \in K : f(x,y) < 0\}$$

is nonempty, which is also convex by hypothesis (b) of the theorem. Again for each $y \in K$, since

$$F^{-1}(y) = \{x \in K : f(x,y) < 0\},$$

it follows that

$$\begin{aligned}[F^{-1}(y)]^C &= \{x \in K : f(x,y) \geq 0\} \\ &\subset \{x \in K : f(y,x) \leq 0\} \quad \text{(by (d))} \\ &= G(y) \quad \text{(say)}.\end{aligned}$$

Thus

$$F^{-1}(y) \supset [G(y)]^C = A(y) \text{ (say)}.$$

Since for each $x \in K$, the map $y \mapsto f(x,y)$ is lower semicontinuous, it follows that $G(y)$ is closed in K. Thus for each $y \in K$, $A(y)$ is open in K.

Now, let $x \in K$ be arbitrary. By (8) there exists $v \in K$ such that $f(v,x) > 0$, so that $x \in A(v)$. Thus $\bigcup_{y\in K} A(y) = K$. Finally, $D = \bigcap_{y\in K_0} G(y) = \bigcap_{y\in K_0} [A(y)]^C$ is either empty or compact by hypothesis (e), so that all the conditions of Theorem 2.9 are fulfilled by F. Thus there exists $x_0 \in K$ such that $x_0 \in F(x_0)$, which is tantamount to $f(x_0, x_0) < 0$, a contradiction to hypothesis (a) of the theorem. Hence the conclusion of the theorem follows.

4 Uniqueness of Solutions

The following theorem characterizes the uniqueness of solution of the Problem P_1 under certain conditions.

4.1 Theorem

Let K be a nonempty subset of a topological vector space X and let $f: K \times K \to \mathbb{R}$ a map such that

(a) *for all $x, y \in K, f(x,y) + f(y,x) \leq 0$,*
(b) *equality does not hold in* (a) *unless $x = y$.*

Then, if Problem P_1 *is solvable, it has a unique solution.*

Proof. If possible let x_1, x_2 be two solutions of P_1. Then

$$f(x_1, y) \geq 0$$

and

$$f(x_2, y) \geq 0$$

for all $y \in K$. Putting $y = x_1$ in the former inequality and $y = x_2$ in the latter, we get

$$f(x_1, x_2) \geq 0$$

and

$$f(x_2, x_1) \geq 0.$$

Adding the last two inequalities we get,

$$f(x_1, x_2) + f(x_2, x_1) \geq 0.$$

This inequality combined with hypothesis (a) gives

$$f(x_1, x_2) + f(x_2, x_1) = 0.$$

Now an application of hypothesis (b) yields $x_1 = x_2$. This completes the proof.

Note: If we set $f(x,y) = (Tx, y - x)$ for some operator $T: K \to X^*$ then condition (a) of the above theorem is equivalent to the fact that T is monotone.

The following examples illustrate Theorem 4.1. Example 1 shows that the fulfillment of conditions (a) and (b) does not guarantee the existence of a solution to problem P_1. Examples (2) and (3) show that if conditions (a) and (b) are fulfilled and problem P_1 is solvable, then it has a unique solution.

Example 1. Let $X = \mathbb{R}$, $K = [0, \infty)$, and define $f: K \times K \to \mathbb{R}$ as

$$f(x,y) = -|x - y|.$$

Then

$$f(x,y) + f(y,x) = -2|x - y|$$

and

$$f(x,y) + f(y,x) = 0$$

if $|x - y| = 0$, that is $x = y$. Thus hypotheses (a) and (b) of Theorem 4.1 are fulfilled by f. But $f(x_0, y) \geq 0$forall $y \in K$ is equivalent to $|x_0 - y| \leq 0$ for all $y \geq 0$. Thus in this case no solution exists.

Example 2. Let $X = \mathbb{R}$, $K = [0, \infty)$, and define $f: K \times K \to \mathbb{R}$ as

$$f(x,y) = x(y - x).$$

Then

$$f(x,y) + f(y,x) = -(x - y)^2 \leq 0$$

and

$$f(x,y) + f(y,x) = 0$$

if and only if $(x - y)^2 = 0$ i.e.$x = y$. Furthermore, $f(x_0, y) \geq 0$ forall $y \in K$ is equivalent to $x_0(y - x_0) \geq 0$ for all $y \geq 0$, so that $x_0 = 0$ is the only solution in this case.

Example 3. Let$X = K = \mathbb{R}$, and define $f: K \times K \to \mathbb{R}$ as

$$f(x,y) = -x^2|x - y|.$$

Then

$$f(x,y) + f(y,x) = -(x^2 + y^2)|x - y| \leq 0$$

and

$$f(x,y) + f(y,x) = 0$$

if and only if $|x - y| = 0$ i.e. $x = y$. Furthermore, $f(x_0, y) \geq 0$ for all $y \in K$ is equivalent to $x_0^2|x_0 - y| = 0$ forall $y \in \mathbb{R}$ so that $x_0 = 0$ is the only solution in this case also.

5 Some Consequences

In this section, we discuss some consequences of the theorems proved in Section 3.

The following theorem deals with the existence of solution of Problem P_2.

5.1 Theorem

Let X be a locally convex Hausdorff topological vector space with *dual X^*,K a nonempty compact convex subset of $X, f: K \times K \to \mathbb{R}$ a point-to-point map, and $S : K \to 2^K$,a point-to-set upper semicontinuous map such that*

a) $f(x,x) \geq 0$ *for each* $x \in K$,
b) *the map* $y \mapsto f(x,y)$ *of* K *into* $\mathbb{R}$ *is quasi convex for each* $x \in K$,
c) *the set* $\left\{x \in K: \inf_{y \in S(x)} f(x,y) < 0\right\}$ *is empty or open in* K.

Then there exists $x_0 \in K$ such that $x_0 \in S(x_0)$ and $f(x_0, y) \geq 0$ for all $y \in S(x_0)$.

Proof. Assume that the assertion is false. Then for each $x \in K$ either $x \notin S(x)$, or there exists $y \in S(x)$ such that $f(x,y) < 0$. If $x \notin S(x)$, then by Theorem 2.11, there exists $p \in X^*$ such that

$$(p,x) - \sup_{y \in S(x)} (p,y) > 0.$$

Let

$$V_0 = \left\{x \in K: \inf_{y \in S(x)} f(x,y) < 0\right\},$$

and for each $p \in X^*$ let

$$V(p) = \left\{x \in K: (p,x) - \sup_{y \in S(x)} (p,y) > 0\right\}.$$

Then by hypothesis (c), V_0 is open in K. By Theorem 2.6, since the map $x \longmapsto \sup_{y \in S(x)}(p,y)$ is upper semicontinuous for each$p \in X^*$, it follows that the map $x \longmapsto (p,x) - \sup_{y \in S(x)}(p,y)$ is lower semicontinuous. Hence $V(p)$ is open in K for each $p \in X^*$. Further, $K = V_0 \cup \left[\bigcup_{p \in X^*} V(p)\right]$. Since K is compact, there exists $\{p_1, p_2, \cdots, p_n\} \subset X^*$ such that $K = V_0 \cup [\bigcup_{i=1}^{n} V(p_i)]$. Let $\{\beta_0, \beta_1, \cdots, \beta_n\}$ be a partition of unity subordinate to the covering $\{V_0, V(p_1), \cdots, V(p_n)\}$, and define $g: K \times K \to \mathbb{R}$ as

$$g(x,y) = \beta_0(x) f(x,y) + \sum_{i=1}^{n} \beta_i(x)(p_i, y - x).$$

Then it is clear that for each $x \in K$, $g(x,x) \geq 0$, the map $y \longmapsto g(x,y)$ is quasi-convex, for each $y \in K$, the map $x \longmapsto g(x,y)$ is upper semicontinuous so that the set $\{x \in K: g(x,y) < 0\}$ is open in K. Thus all the conditions of Theorem 3.1 are fulfilled. Hence there exists $\hat{x} \in K$ such that

$$g(\hat{x}, y) \geq 0 \tag{9}$$

for all $y \in K$, which is equivalent to

$$\beta_0(\hat{x})f(\hat{x},y)+\sum_{i=1}^{n}\beta_i(\hat{x})(p_i,y-\hat{x})\geq 0.$$

Since $\{\beta_0,\beta_1,\cdots,\beta_n\}$ is a partition of unity $\beta_i(\hat{x})>0$ for at least one i. If $\beta_0(\hat{x})>0$ then $\hat{x}\in V_0$ so that in $\mathrm{f}_{y\in S(\hat{x})}f(\hat{x},y)<0$. Let $\hat{y}\in S(\hat{x})$ be such that $f(\hat{x},\hat{y})<0$. If $\beta_i(\hat{x})>0$ then $\hat{x}\in V(p_i)$ so that

$$(p_i,\hat{x})-\sup_{y\in S(\hat{x})}(p_i,y)>0$$

from which it follows that

$$(p_i,\hat{x})>\sup_{y\in S(\hat{x})}(p_i,y)>(p_i,\hat{y})$$

which implies that $(p_i,\hat{y}-\hat{x})<0$. Thus,

$$g(\hat{x},\hat{y})=\beta_0(\hat{x})f(\hat{x},\hat{y})+\sum_{i=1}^{n}\beta_i(\hat{x})(p_i,\hat{y}-\hat{x})<0$$

which contradicts (9) when $y=\hat{y}$. This contradiction proves the theorem.

The following result on the existence of solution of the quasi-variational inequality problem is a direct consequence of the above theorem.

5.2 Corollary

Let X be a locally convex Hausdorff topological vector space with dual X^*, K a nonempty compact convex subset of X, $T:K\to X^*$ an operator and $S:K\to 2^K$ a point-to-set upper semi-continuous map such that the set $\{x\in K:\inf_{y\in S(x)}(Tx,y-x)<0\}$ is empty or open in K. Then there exists $x_0\in K$ such that $x_0\in S(x_0)$ and $(Tx_0,y-x_0)\geq 0$ for all $y\in S(x_0)$.

Proof. Follows directly from Theorem 5.1 with $f(x,y)=(Tx,y-x)$.

6 Conclusion

The variational inequality technique is a powerful technique for handling a wide range of problems arising in diversified areas of since and engineering. Problem P_1, which is a variant of the equilibrium problem,is a generalization to the classical variational inequality problem and some of its generalizations. Problem P_1 can be studied in a more general setting, for example, by considering X an H-space.

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Authors have declared that no competing interests exist.

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