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A Hybrid of the New Conjugate Gradient Method and Galerkin Theory for Optimizing Beam Deflection under Uniformly Distributed Load

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Abstract

A hybrid of the new Conjugate gradient method and Galerkin theory has been used to find the maximum deflection of a beam under uniformly distributed load. Maximum deflection of a beam under a given pressure was found by solving a two-point linear, second order, boundary value problem with homogeneous boundary conditions without evaluating the inverse of a matrix. An objective function associated with a given member of this class of boundary value problems was optimized. The numerical results obtained from solving some of these problems are very close to the exact solutions. This method is easy to implement and automate computer-wise.

Keywords: Hybrid of the new conjugate gradient method, two-point linear boundary value problems, objective function, Galerkin theory.

1 Introduction

Burden and Faires [1] have shown how physical problems that are position- dependent rather than time-dependent could be described in terms of differential equations. A differential equation of

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this type has conditions imposed at more than one point. A common problem in civil engineering is to find the maximum deflection of a beam subject to uniform loading while the ends are supported in order to avoid deflection at the two fixed ends. A mathematical model of this physical phenomenon results in the formulation of a boundary value problem (BVP). A BVP is an ordinary differential equation with specified values at the extreme points or boundaries of a given system [2]. A beam deflection model, formulated from Fig. 1, seeks the value of a function $y(x)$ from the following two-point linear boundary value problem.

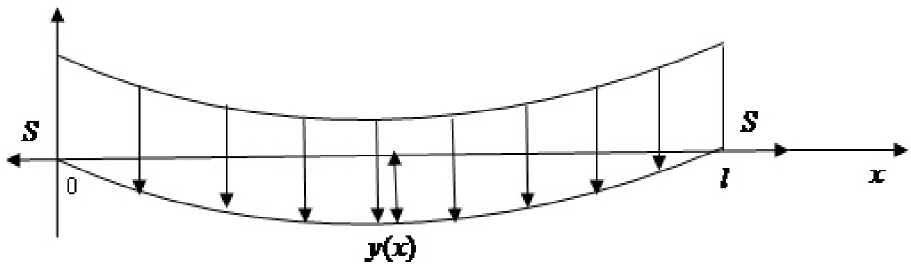

Fig. 1. Simply supported beam with a uniformly distributed load

$$\frac{d^2y(x)}{dx^2} = \frac{S}{EI}y(x) + \frac{q}{2EI}x(l-x)\,, 0 \le x \le l \tag{1}$$

$$y(0) = 0 = y(l),$$

where x is the location along the beam, l is the length of the beam, E is the young's modulus of elasticity of the beam, I is the central moment of inertia, S is the stress at the endpoints and q is the intensity of the uniform load. In a related literature, Mohammadi et al. [3] employed the differential quadrature method and Galerkin method in their investigation of free vibration behavior of rectangular graphene sheet under shear in-plane load. Also, Ali et al. [4] used the modified Timoshenko beam model to derive a formulation which provides more accurate results than those obtained by the classical beam theory. In addition, Mohammad et al. [5] used the differential quadrature method (DQM) to solve the governing equations of the nanorod for clamped–clamped (C–C), clamped–free (C–F) and fixed-attached spring boundary conditions. This paper will concentrate on linear two-point, second order, boundary value problem given in equation (1). This type of boundary value problem is assumed to have a unique solution, $y(x)$, since $\frac{q}{2E}x(l-x)$ is continuous in the given interval. In sections (2) and (3), we considered other relevant literatures and Galerkin method for solving two-point linear, second order, boundary value problems. Sections (4) and (5) treated the new conjugate gradient method with Galerkin theory. Numerical examples and solutions were considered in sections (6) and (7). Section (8) discussed our numerical results. Finally, section (9) summarized the findings of this paper with a conclusion.

2 Literature Review

Numerical methods have been used to generate an approximate solution of the conventional linear two-point, second order, boundary value problem since the analytic solution is very difficult to handle. This problem takes the form

$$d(x)y''(x)+e(x)y'(x)+v(x)y(x)=t(x),\ [a,b]. \tag{2}$$

A unique solution of equation (2) exists if $d(x),\ e(x),\ v(x)$ and $t(x)$ are continuous in $[a,b]$ [1,6,7]. Shooting method and finite difference method have been used to solve this class of problems [1,6,7,8]. Fyfe, in 1968 introduced cubic spline interpolation method for solving equation (2). After Fyfe, many researchers, including Burden and Faires, used linear and cubic spline interpolations with Galerkin method and finite element method to solve same problem [1,9,10]. Galerkin method is one of the best methods for solving (2) numerically. The numerical solutions of this class of boundary value problems are very good but difficult to implement. The method leads to full matrices that must be inverted in order to obtain the required solutions [11]. Shafigul and Shirin [12] used Galerkin method with Bernoulli polynomials to solve equation (2). Also, Rahman et al. [13] used Galerkin method with hermite polynomials to solve same problem. Bamigbola and Ejieji [14] showed that the properties of an objective function could be explored to design an efficient conjugate gradient method for solving optimization problems. The new conjugate gradient method [15] has been used to solve many optimization problems successfully without involving matrix inversion. We, therefore, present a hybrid of the new conjugate gradient method and Galerkin theory for solving problem (1) with ease and high accuracy.

3 Galerkin Method

Galerkin method is a variation technique used in solving equation (2) numerically. In this paper, it is based on the fact that a function $w\in C^2[0,1]$ is the unique solution of $-\left[d(x)y'(x)\right]' + v(x)y(x) = t(x),$ for $0\le x\le 1,$ if and only if *w* is a unique function in $C^2[0,1]$ whenever

$$\int_0^1 \left\{-\left[d(x)w'(x)\right]' + v(x)w(x) - t(x)\right\}\phi_j(x)dx = 0 \tag{3}$$

$i=1,\ldots,n.$ From equation (1), $v(x) = -\dfrac{S}{EI},\ t(x) = \dfrac{q}{2EI}x(l-x),\ d(x) = -1$ and ϕ_j is a weight function. Equation (3) yields n residual equations in n unknowns. This method approximates the solution of (1) by solving the system of equations derived from equation (3) simultaneously. In order to solve this system of equations, we used the finite element method as a variation tool. The finite element method is simply the Ritz-Galerkin method where the finite set of basis functions $\phi_1(x),\ \phi_2(x),\ \ldots,\ \phi_n(x)$ are splines. With these splines, the minimum of equation (3) is usually computed. We restrict ourselves to the splines defined in equation (6).

Thus, we convert the interval, $[0, l]$ in equation (1), to $[0, 1]$ by the relation $x \to z$ such that $z = lx,\ x \in [0, 1]$ and $z \in [0,\ l]$. It follows that $f(x) = t(z),\ p(x) = \frac{d(z)}{l^2} = -\frac{1}{l^2}$, $r(x) = v(z)$ and $y(x) = w(z)$. $p(x)$ becomes the new coefficient of y''. From equation (3),

$$\int_0^1 \left\{-\left[p(x)y'(x)\right]' + r(x)y(x) - f(x)\right\}\phi_j(x)dx = 0$$

$$\int_0^1 \left\{-\left[p(x)y''(x) + p'(x)y'(x)\right] + r(x)y(x) - f(x)\right\}\phi_j(x)dx = 0$$

$$\int_0^1 p(x)\phi_j y''(x)dx = y'(x)p(x)\phi_j(x)\Big|_0^1 - \int_0^1 y'\left(p'(x)\phi_j(x) + p(x)\phi_j'(x)\right)dx$$

$$= -\int_0^1 y'\left(p'(x)\phi_j(x) + p(x)\phi_j'(x)\right)dx$$

It follows that

$$\int_0^1 \left\{p(x)\phi_j'(x)y'(x) + r(x)y(x)\phi_j(x)\right\}dx = \int_0^1 f(x)\phi_j(x)dx \tag{4}$$

We partition the interval $[0, 1]$ into $n+1$ subintervals with n interior points such that $0 = x_0 < x_1 < \ldots < x_n < x_{n+1} = 1; h = x_{i+1} - x_i; x_i = ih; i = 0, 1, 2, \ldots, n+1.$

A small set of functions which consists of linear combinations of some basis functions $\phi_1(x), \phi_2(x), \ldots, \phi_n(x)$ contains the approximate solutions of (1) [16]. These functions are required to be linearly independent and satisfy the conditions given in equation (5) below.

$$\phi_i(0) = \phi_i(1) = 0; i = 1, 2, \ldots, n. \tag{5}$$

In this paper, we used the conventional basis functions and their derivatives as defined in equations (6) and (7) below.

$$\phi_i(x) = \begin{cases} 0; & 0 \le x \le x_{i-1} \\ \dfrac{x - x_{i-1}}{h}; & x_{i-1} < x \le x_i \\ \dfrac{x_{i+1} - x}{h}; & x_i < x \le x_{i+1} \\ 0; & x_{i+1} < x \le 1 \end{cases} \tag{6}$$

$$\phi_i'(x) = \begin{cases} 0; & 0 < x < x_{i-1} \\ \frac{1}{h}; & x_{i-1} < x < x_i \\ -\frac{1}{h}; & x_i < x < x_{i+1} \\ 0; & x_{i+1} < x < 1 \end{cases} \tag{7}$$

$i = 1, ..., n$, $\phi_i(x_i) = 1,\ \phi_i(x_{i-1}) = 0$ and $\phi_i(x_{i+1}) = 0$. If we connect (x_i, c_i) by a line segment we obtain an approximation of the form

$$y_n(x) = \sum_{i=1}^{n} c_i \phi_i(x) \tag{8}$$

where c_i approximates the exact solution of (1) at $y(lx_i)$. It follows that

$$y_n'(x) = \sum_{i=1}^{n} c_i \phi_i'(x) \tag{9}$$

The absolute error of approximation is $|y_n(x) - y(lx)|$. Replacing y by $y_n(x)$ and y' by $y_n'(x)$ in equation (4) gives

$$\int_0^1 \{p(x) y_n'(x)\phi_j'(x) + r(x) y_n(x)\phi_j(x)\} dx = \int_0^1 f(x)\phi_j(x)dx$$

Substitute for $y_n(x)$ and $y_n'(x)$ values, from equations (8) and (9), into above equation:

$$\int_0^1 \left\{\left[p(x)\sum_{i=1}^{n} c_i\phi_i'(x)\right]\phi_j'(x) + r(x)\left[\sum_{i=1}^{n} c_i\phi_i(x)\right]\phi_j(x)\right\}dx = \int_0^1 f(x)\phi_j(x)dx$$

$$\sum_{i=1}^{n}\left[\int_0^1 \{p(x)\phi_i'(x)\phi_j'(x) + r(x)\phi_i(x)\phi_j(x)\}dx\right]c_i = \int_0^1 f(x)\phi_j(x)dx \tag{10}$$

The solution of equation (10) will produce $n \times n$ linear system of equations in $c_i, i = 1, 2, ..., n$, unknown constants. So, we must form and solve a matrix equation. Define

$$a_{i,j} = \int_0^1 \{p(x)\phi_i'(x)\phi_j'(x) + r(x)\phi_i(x)\phi_j(x)\}dx \tag{11}$$

$$b_j = \int_0^1 f(x)\phi_j(x)dx \tag{12}$$

From equation (11),

$$
\begin{aligned}
a_{i,i} &= \int_0^1 \left\{ p(x)\left[\phi_i'(x)\right]^2 + r(x)\left[\phi_i(x)\right]^2 \right\} dx \\
&= \int_{x_{i-1}}^{x_i} \left\{ p(x)\left[\phi_i'(x)\right]^2 + r(x)\left[\phi_i(x)\right]^2 \right\} dx + \int_{x_i}^{x_{i+1}} \left\{ p(x)\left[\phi_i'(x)\right]^2 + r(x)\left[\phi_i(x)\right]^2 \right\} dx \qquad (13) \\
&= \frac{1}{h^2}\int_{x_{i-1}}^{x_i} \left\{ p(x) + r(x)\left[x - x_{i-1}\right]^2 \right\} dx + \frac{1}{h^2}\int_{x_i}^{x_{i+1}} \left\{ p(x) + r(x)\left[x_{i+1} - x\right]^2 \right\} dx
\end{aligned}
$$

$i = 1, 2, ..., n.$

$$a_{i,i+1} = \int_{x_i}^{x_{i+1}} \left\{ p(x)\phi_i'(x)\phi_{i+1}'(x) + r(x)\phi_i(x)\phi_{i+1}(x) \right\} dx \qquad (14)$$

$$= \frac{1}{h^2}\int_{x_i}^{x_{i+1}} \left\{ -p(x) + r(x)\left[x - x_i\right]\left[x_{i+1} - x\right] \right\} dx$$

$i = 1, 2, ..., n-1.$

$$a_{i,i-1} = \int_{x_{i-1}}^{x_i} \left\{ p(x)\phi_i'(x)\phi_{i-1}'(x) + r(x)\phi_i(x)\phi_{i-1}(x) \right\} dx \qquad (15)$$

$$= \frac{1}{h^2}\int_{x_{i-1}}^{x_i} \left\{ -p(x) + r(x)\left[x - x_{i-1}\right]\left[x_i - x\right] \right\} dx$$

$i = 2, ..., n.$

From equation (12),

$$b_j = \int_{x_{j-1}}^{x_j} \left\{ f(x)\phi_j(x) \right\} dx + \int_{x_j}^{x_{j+1}} \left\{ f(x)\phi_j(x) \right\} dx \qquad (16)$$

$$= \frac{1}{h}\int_{x_{j-1}}^{x_j} \left\{ f(x)\left(x - x_{j-1}\right) \right\} dx + \frac{1}{h}\int_{x_j}^{x_{j+1}} \left\{ f(x)\left(x_{j+1} - x\right) \right\} dx$$

$j = 1, 2, ..., n.$

From the above equations, we obtained the matrix equation for the system:

$$AC = B \qquad (17)$$

where

$$A = a_{ij}, \quad B = b_j, \quad C = \begin{pmatrix} c_1 \\ c_2 \\ . \\ . \\ . \\ c_n \end{pmatrix}, \quad j = 1, 2, ..., n \text{ and } i = 1, 2, ..., n.$$

From equation (17),

$$C = A^{-1}B \tag{18}$$

The absolute error is: Error = $\left| y_n(x) - y(lx) \right|$.

4 New Conjugate Gradient Method

The new conjugate gradient method (NCGM) [13] seeks to optimize a multivariable function f whose gradient vector is g. At a point x_k, the objective function F is represented by

$$F(x) = f(x_k) + wf(x_k) + \frac{1}{2}w^2 f(x_k) + \ldots + \frac{1}{m!}w^m f(x_k)$$

where

$$w^m F(x_k) = \sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\ldots\sum_{i_N=1}^{N} h_{i_1}h_{i_2}\ldots h_{i_N}\frac{\partial^m f(x_k)}{\partial x_{i_1}\partial x_{i_2}\ldots\partial x_{i_N}},\ x, h \in \Re^m,\ h_j = x_j - x_{j-i};\ n \geq 2.$$

The gradient function of F is represented by G. The algorithm for the scheme is given below.

4.1 Algorithm I (NCGM)

Input initial values: x_0 and $D_0 = -G_0$.

Repeat:

Find step length α_k such that $F(x_k + \alpha_k D_k = \min F(x_k + \alpha D_k)$.
Compute new point: $x_{k+1} = x_k + \alpha_k D_k$. Update search direction:

$$D_{k+1} = -G_{k+1} + \beta_k D_k,\ G_{k+1} = \frac{1}{2}\left[g(x_k + 2\Delta x_k) + g(x_k)\right],\ \beta_k = \frac{\| G_{k+1} \|^2}{D_k^{\ T} u_k}$$

$u_k = G_{k+1} - G_k$. $\|\,.\,\|$ denotes Euclidean norm.

Check for optimality of g : Terminate iteration at step k if $\| g(x_k) \|$ is so small that x_k is acceptable.

5 New Conjugate Gradient Method with Galerkin Theory

The hybrid of the new Conjugate Gradient Method and Galerkin theory seeks to $\underset{0 \le x \le l}{Maximize}\ y(x)$ subject to equation (1). By our method, we require the matrix A, in equation (16), to be symmetric otherwise we replace A with an equivalent symmetric matrix M. That is, $A^T AC = A^T B$

or

$$MC = Z \tag{19}$$

where $M = A^T A$ and $Z = A^T B$. Also, $M = A$ if A is symmetric. By our assumption of uniqueness of solution of (1), we state that the matrix M is also positive definite. It follows that we are guaranteed to form an objective function from equation (19) as given below.

$$F(C) = \frac{1}{2} C^T MC - C^T Z \tag{20}$$

The gradient function is

$$G(C) = MC - Z \tag{21}$$

Next, we seek to solve the following optimization problem.

$$Minimize\ (-F\ )\ over\ C \tag{22}$$

Since our objective function F in equation (20) is a quadratic function, we used the new Conjugate Gradient Method to solve (22). This technique solves the optimization method through an iterative procedure

$$\mathrm{C}_{\mathrm{k+1}} = C_k + \alpha_k d_k \tag{23}$$

$k = 0, 1, 2, \ldots,$ d_k is a direction vector and α_k is the line search step length at iteration k. Usually, we find α_k such that $\alpha_k = \min_{\alpha > 0} (-F(C_k + \alpha d_k))$. The algorithm is given below.

5.1 Algorithm II (Hybrid of the New Conjugate Gradient Method and Galerkin Theory)

Let $G_k = G(C_k)$ and $\|G_k\| = \left(G_k{}^T G_k\right)^{\frac{1}{2}}$. Choose a whole number $n\ and\ h = \dfrac{1}{n+1}$. Use Galerkin method to obtain equations (17) and (19) and $G(C_k) = MC_k - Z$ from equation (21). Use the following initial values:

$$C_0 = (0, 0, ..., 0)^T, \; k = 0, \; D_0 = -G_0.$$

Repeat the following steps of the new Conjugate gradient method.

a. Find step length α_k such that

$$F(C_k + \alpha_k D_k) = \min_{\alpha > 0} F(C_k + \alpha D_k)$$

b. Compute new point:

$$C_{k+1} = C_k + \alpha_k D_k$$

c. Update search direction:

$$D_{k+1} = -G_{k+1} + \beta_k D_k,$$

$$G_{k+1} = \frac{1}{2}\left[G(C_k + 2\Delta C_k) + G(C_k)\right]$$

$$\beta_k = \frac{\|G_{k+1}\|^2}{D_k^{\,T} y_k}$$

$$y_k = G_{k+1} - G_k$$

d. Check for optimality of G : Terminate iteration at step k when $\|G_k\|$ is so small that C_k is an acceptable estimate of the optimal point of F . If not optimal set $k = k + 1$.

6 Numerical Examples

We used the hybrid of the new Conjugate Gradient Method and Galerkin theory to find the maximal deflection of a simply supported beam under uniformly distributed load from the following boundary value problems. Each boundary value problem governs the deflection of a structured beam as described in exercise 11.3, page 666, of [1]. Nine interior points were used in each case.

Problem 1: The boundary-value problem governing the deflection of a beam, in Fig. 1, with supported ends, is

$$\frac{d^2 y(x)}{dx^2} = \frac{S}{EI} y(x) + \frac{q}{2EI} x(120 - x), \quad 0 \le x \le 120$$
$$y(0) = 0 = y(120).$$

Suppose the beam is a W10-type steel I-beam with the following characteristics:

$$S = 1000N, \; l = 120m, \; q = 8\frac{1}{3} N/m, \; E = 3\times10^7 N/m^2 \; and \; I = 625m^4;$$

where l is the length, S is the stress at the ends, q is the intensity of uniform load, E is the modulus of elasticity and I is the central moment of inertia. Find the maximum deflection of the beam at its middle.

Solution of problem (1): In the interval $[0, 1]$, we restated problem (1) as

$$-\left[p(x)y'(x)\right]' + r(x)y(x) = f(x), \quad y(0) = 0 = y(1), \ \ 0 \le x \le 1;$$

where $p(x) = -\dfrac{1}{l^2} = -\dfrac{1}{14400}$, $r(x) = -\dfrac{S}{EI}$ *and* $f(x) = \dfrac{q}{2EI}120x(120-120x).$

Algorithm (5.1) was used to solve the problem. The solution is shown in Table 1.

Table 1. Solution of problem 1 (Maximal deflection is 0.12cm downwards)

$n = 9, h = \frac{1}{10}, \ x_i \in [0,1]$				
i	x_i	$y_9(x_i) = c_i$	$y(120x_i)$	**Error =** $\lvert y_9(x_i) - y(120x_i)\rvert$
1	0.1	-0.000376675239629	-0.000376675016014	2.23615×10^{-10}
2	0.2	-0.000712649320008	-0.000712648849003	4.71005×10^{-10}
3	0.3	-0.000975668780120	-0.000975668110186	6.69934×10^{-10}
4	0.4	-0.001142695610452	-0.001142694818554	7.91898×10^{-10}
5	**0.5**	**-0.001199907076070**	**-0.001199906197144**	**8.78927×10^{-10}**
6	0.6	-0.001142695610458	-0.001142694702139	9.08319×10^{-10}
7	0.7	-0.000975668780131	-0.000975667935563	8.44568×10^{-10}
8	0.8	-0.000712649320025	-0.000712648557965	7.6206×10^{-10}
9	0.9	-0.000376675239651	-0.000376674608560	6.3109×10^{-10}

Exact solution of problem (1): The deflection of the beam at point x is given by

$$y(x) = C_1 e^{x\sqrt{a}} + C_2 e^{-x\sqrt{a}} + \frac{b}{a}x^2 - \frac{120b}{a}x + \frac{2b}{a^2}, \quad 0 \le x \le 120.$$

Where

$C_1 = -77042.53752198143, \ C_2 = -79207.46247801857, \ a = 5.33333 \times 10^{-8}$ *and*
$b = 2.22222222 \times 10^{-10}$

Problem 2: The boundary-value problem governing the deflection of a $20m$ long beam with flexural stiffness EI is given by

$$\frac{d^2y(x)}{dx^2} = \frac{q}{2EI}x(20-x), \quad 0 \le x \le 20$$
$$y(0) = 0 = y(20),$$

$l = 20m,\ q = 5000N/m,\ EI = 133.333\times10^6\,Nm^2$ *and* q is the intensity of the uniform load. Find the maximum deflection of the beam at its middle.

Solution of problem (2): In the interval $[0, 1]$, we restated problem (2) as

$$-[p(x)y'(x)]' + r(x)y(x) = f(x), \quad y(0) = 0 = y(1), \ 0 \le x \le 1;$$

where $p(x) = -\frac{1}{l^2} = -\frac{1}{400}$, $r(x) = 0$ *and* $f(x) = \frac{q}{2EI}20x(20-20x)$.

Algorithm (5.1) was used to solve the problem. The solution is shown in Table 2.

Table 2. Solution of problem 2 (Maximal deflection is 7.81cm downwards)

$n = 9, h = \frac{1}{10}, \ x_i \in [0,1]$				
i	x_i	$y_9(x_i) = c_i$	$y(20x_i)$	**Error =** $\lvert y_9(x_i) - y(20x_i)\rvert$
1	0.1	-0.024525000061357	-0.024525000061312	4.4×10^{-14}
2	0.2	-0.046400000116089	-0.046400000116000	8.9×10^{-14}
3	0.3	-0.063525000158946	-0.063525000158812	1.33×10^{-13}
4	0.4	-0.074400000186200	-0.074400000186000	1.78×10^{-13}
5	**0.5**	**-0.078125000195535**	**-0.078125000195312**	$\mathbf{2.22 \times 10^{-13}}$
6	0.6	-0.074400000186266	-0.074400000186000	2.66×10^{-13}
7	0.7	-0.063525000159123	-0.063525000158812	3.11×10^{-13}
8	0.8	-0.046400000116355	-0.046400000116000	3.55×10^{-13}
9	0.9	-0.024525000061712	-0.024525000061312	4×10^{-13}

Exact solution of problem (2): The deflection of the beam at point x is given by

$$y(x) = \frac{b}{12}\left(40x^3 - x^4 - 8000x\right), \ 0 \le x \le 20.$$
$$b = \frac{q}{2EI}.$$

Problem 3: The deflection of a uniformly loaded, long rectangular plate under an axial tension force is governed by a second-order differential equation. Let S represent the axial force and q the intensity of the uniform load. The deflection w along the elemental length is given by

$$\frac{d^2y(x)}{dx^2} = \frac{S}{D}y(x) + \frac{q}{2D}x(50-x), \quad 0 \le x \le 50$$
$$y(0) = 0 = y(50),$$

$$S = 100N, \; l = 50m, \; q = 200N/m, \; D = 8.8\times 10^7 Nm^2;$$

where l is the length of the plate and D is the flexural rigidity of the plate. Find the maximum deflection of the beam at its middle.

Solution of problem (3): In the interval $[0, 1]$, we restated problem (3) as

$$-[p(x)y'(x)]' + r(x)y(x) = f(x), \quad y(0) = 0 = y(1), \; 0 \le x \le 1;$$
where $p(x) = -\frac{1}{l^2} = -\frac{1}{2500}$, $r(x) = -\frac{S}{D}$ *and* $f(x) = \frac{q}{2D}50x(50-50x).$

Algorithm (5.1) was used to solve the problem. The solution is shown in Table 3.

Exact solution of problem (3): The deflection of the beam at point x is given by

$$y(x) = C_1e^{x\sqrt{a}} + C_2e^{-x\sqrt{a}} + x^2 - 50x + \frac{2}{a}, \quad 0 \le x \le 50,$$
$$C_1 = -856553471726025, \; C_2 = -9034465282739757 \; and$$
$$a = 1.13636363636363636\times 10^{-6}$$

Table 3. Solution of Problem 3 (Maximal deflection is 18.49cm downwards)

$n=9, h=\frac{1}{10}, \; x_i \in [0,1]$				
i	x_i	$y_9(x_i) = c_i$	$y(50x_i)$	**Error =** $\lvert y_9(x_i) - y(50x_i)\rvert$
1	0.1	-0.058044685436589	-0.058044549310580	1.36126009 x 10^{-7}
2	0.2	-0.109817315930269	-0.109817057382315	2.58547953 x 10^{-7}
3	0.3	-0.150347747852438	-0.150347393937409	3.53915029 x 10^{-7}
4	0.4	-0.176085979463466	-0.176085565239191	4.14224275 x 10^{-7}
5	**0.5**	**-0.18490205005989**	**-0.184901615371928**	**4.34687963 x 10^{-7}**
6	0.6	-0.176085979463886	-0.176085566170514	4.13293373 x 10^{-7}
7	0.7	-0.150347747853279	-0.150347396032885	3.51820394 x 10^{-7}
8	0.8	-0.10981731593153	-0.109817060641944	2.55289585 x 10^{-7}
9	0.9	-0.05804468543827	-0.058044552803040	1.32635231 x 10^{-7}

7 Numerical Solutions

We used MatLab to solve the given boundary value problems.

8 Discussion of Numerical Results

The data shown in Tables 1 to 3 reveal that results obtained by the hybrid of the new conjugate gradient method and Galerkin theory are very close to the exact solutions of the given boundary value problems. We observed that errors on numerical solutions decrease as interior points increase.

9 Conclusion

We have used this hybrid of a new conjugate gradient method and Galerkin theory to find the maximal deflection value of a simply supported beam under uniformly distributed load. The characterized two-point linear, second order, boundary value problems with homogeneous boundary conditions were solved, numerically. Simple basis functions were used as trial functions in our approximations. Results from Tables 1 to 3 confirm that our method is accurate and reliable.

Herein, we present the hybrid of a new conjugate gradient method and Galerkin theory to engineers and scientists who wish to solve real life problems in this class of boundary value problems.

Competing Interests

Authors have declared that no competing interests exist.

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