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A Design Method of Variable Gain Robust Controllers with Guaranteed $\mathcal{L}_2$ Gain Performance for a Class of Uncertain Switched Linear Systems

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Abstract

This paper discusses a design problem of a variable gain robust controller with guaranteed $\mathcal{L}_2$ gain performance for a class of uncertain switched linear systems. The uncertainties included in the switched linear system under consideration are supposed to satisfy the matching condition and the proposed variable gain robust controller consists of a switching rule, state feedback laws with fixed and variable feedback gain matrices. The switching rule and the fixed feedback gain matrices are derived by using the nominal system. In this paper, we show that a design method of the variable gain robust controller with guaranteed $\mathcal{L}_2$ gain performance are reduced to matrix inequalities. Besides, it is presented that the number of matrix inequalities in the proposed design is less than one for the existing results. Finally, an illustrative example is included.

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1 Introduction

Robust control is very important topic in the control engineering community, because unavoidable discrepancies between mathematical models and systems in practice, referred to as "uncertainties", can degrade the performance of control systems. Therefore for dynamical systems with uncertainties lots of researchers have considered robust control problems such as robust stability analysis, robust stabilization and so on (e.g.[1, 2] and references therein). In addition, some design methods of variable gain controllers for uncertain linear systems have also been shown (e.g. [3, 4]). In the work of Maki and Hagino[3] have presented a robust controller with adaptation mechanism for linear systems with time-varying parameter uncertainties. Additionally, we have proposed a robust controller with an adaptive compensation input which achieves not only asymptotical stability but also improving transient behavior[4]. This robust controller is one of variable gain robust controllers which are tuned by updating laws and the adaptive compensation input is designed to reduce the effect of unknown parameters. Besides, input-to-state stability and/or input-to-output performance analysis of dynamical systems have also been well studied in control system theory, and in particular the linear robust $\mathcal{H}^\infty$ control and $\mathcal{L}_2$ gain performance analysis are well-established research area (e.g. [5, 6, 7, 8]).

On the other hand, analysis and controller design for switched systems have received a growing attention in control theory and practice (e.g. [9, 10]). Switched systems which are composed of a family of continuous-time (or discrete-time) subsystems and a switching signal that orchestrates the switching between them are an important class of hyblid systems. Thus switched systems have been well studied from a variety of viewpoints. The first viewpoint is that the switching signals are considered as exogenous variables. The switching signal in this case is not available for controller design, and then the control problem is to investigate whether there exists a switching signal such that the switched system can achieve desired control performance (stability, certain disturbance attenuation level and so on). The second viewpoint, which is of interest in this paper, is that control engineers can utilize the switching rule to achieve satisfactory control performance, i.e. the switching signal can be used for control purposes. In the last three decades, available switchings between subsystems for control purposes have been suggested. In particular, Wicks et al. have established a theoretical strategy based on Lyapunov stability theory[9]. Furthermore, some robust controllers for switched linear systems with uncertainties have also been studied (e.g. [10, 11, 12, 13]). However, these methods have disadvantages such that with the increase of subsystem number of extreme point number, the computation complexity increases and the possibility of quadratic stabilization decreased[11, 12]. From this viewpoint, we have proposed a design method of a variable gain robust controller with adjustable parameters for a class of uncertain switched linear systems, and show that the number of matrix inequalities needed to be solved is always less than the existing results[14, 15].

In this paper on the basis of our works[4, 14] we consider a design problem of a variable gain robust controller with guaranteed $\mathcal{L}_2$ gain performance for a class of uncertain switched linear systems. Namely, we extend our previous works[4, 14] to a variable gain robust controller with guaranteed $\mathcal{L}_2$ gain performance. The variable gain robust controller consists of a switching rule, state feedback laws with constant and variable gain matrices, and uncertainties under consideration are supposed to satisfy the well-known matching condition[16]. The switching rule among subsystems and the fixed gain matrices are determined by using the nominal system. In this paper, the variable gain matrix is also determined in order to compensate the effect of uncertainties. Since the switching rule is determined by using the nominal system, the number of matrix inequalities needed to be solved is less than the conventional robust control with $\mathcal{L}_2$ gain performance based on the existing results (e.g [13]) and thus the proposed controller design method is useful. This paper is organized as follows. In Section 2, we show the notation used in this paper and the well-known existing results. In Section 3, we define a class of uncertain switched linear systems under consideration, and introduce adjustable parameters. Section 4 contains the main results. The design method of the proposed variable gain robust controller with $\mathcal{L}_2$ gain performance can be developed. Finally, a numerical example is presented to illustrate the results developed in this paper.

2 Preliminaries and Existing Results

In this section, we show notations and the well-known existing results for switched linear systems which are used in this paper. Note that the notations and the well-known existing results for switched linear systems which are shown in this section have also been stated in our works[14, 15].

In the sequel, we use the following notation. The transpose of matrix $\mathcal{X}$ and the inverse of one are denoted by $\mathcal{X}^T$ and $\mathcal{X}^{-1}$ respectively. $H_e\{\mathcal{X}\}$ means $\mathcal{X} + \mathcal{X}^T$ and $\mathrm{diag}(\mathcal{X}_1, \cdots, \mathcal{X}_\mathcal{M})$ denotes a block diagonal matrix composed of matrices $\mathcal{X}_i$ for $i = 1, \cdots, \mathcal{M}$. Also, I_n represents n-dimensional identity matrix. For real symmetric matrices $\mathcal{X}$ and $\mathcal{Y}$, $\mathcal{X} > \mathcal{Y}$ (resp. $\mathcal{X} \geq \mathcal{Y}$) means that $\mathcal{X} - \mathcal{Y}$ is positive (resp. nonnegative) definite matrix. Furthermore, for a vector $x \in \Re^n$, $||x||$ denotes standard Euclidian norm and $\|\mathcal{X}\|$ for a matrix $\mathcal{X}$ means a matrix norm induced by the vector norm. The symbols "$\stackrel{\triangle}{=}$" and "$\star$" mean equality by definition and symmetric blocks or symmetric elements in matrices, respectively. Besides, $\mathcal{L}_2[0, \infty)$ is $\mathcal{L}_2$-space (i.e. the collection of all square integrable functions) defined on $[0, \infty)$ and for a signal $f(t) \in \mathcal{L}_2[0, \infty)$, $||f(t)||_{\mathcal{L}_2}$ denotes its $\mathcal{L}_2$-norm.

Next we show the well-known existing results for switched linear systems. Consider the following dynamical system.

$$\frac{d}{dt}x(t) = A_{\sigma(x,t)}x(t) \tag{2.1}$$

where $x(t) \in \Re^n$ is the vector of the state and $\sigma(x, t) \in \mathcal{I}_\mathcal{S} \stackrel{\triangle}{=} \{1, 2, \cdots, \mathcal{S}\}$ is a switching rule. Therefore, the matrix $A_{\sigma(x,t)}$ is allowed to take values only in the set $\{A_1, \cdots, A_\mathcal{S}\}$. Such a dynamical system is said to be "switched linear system" (e.g. [11]). In addition for the switched linear system of (2.1), we show a definition, two theorems for quadratic stabilizability and quadratic stabilization via switchings (e.g. [9, 10, 12, 13]) and two lemmas[17, 18].

Definition 1. [9, 10, 12] The switched linear system of (2.1) is quadratically stabilizable via state feedback switching if there exists a positive definite function $\mathcal{V}(x, t) \stackrel{\triangle}{=} x^T(t)\mathcal{P}x(t)$, a positive number ϵ and a switching rule $\sigma(x, t)$ such that

$$\frac{d}{dt}\mathcal{V}(x, t) < -\epsilon x^T(t)x(t) \tag{2.2}$$

for all trajectories $x(t)$ of the switched linear system of (2.1).

Theorem 1. [10, 12, 13] The switched linear system of (2.1) is quadratically stabilizable via state feedback switching if there exists constant scalars $\tau_k \geq 0$ $(k = 1, \cdots, \mathcal{S})$ with $\sum_{k=1}^{\mathcal{S}} \tau_k = 1$ such that the matrix $\sum_{k=1}^{\mathcal{S}} \tau_k A_k$ is asymptotically stable, i.e. there exist a positive scalar ϵ and a symmetric positive definite matrix $\mathcal{P} \in \Re^{n\times n}$ which satisfy

$$\begin{aligned} H_e\left\{\left(\sum_{k=1}^{\mathcal{S}} \tau_k A_k\right)^T \mathcal{P}\right\} &= \left(\sum_{k=1}^{\mathcal{S}} \tau_k A_k\right)^T \mathcal{P} + \mathcal{P}\left(\sum_{k=1}^{\mathcal{S}} \tau_k A_k\right) \\ &< -\epsilon I_n. \end{aligned} \tag{2.3}$$

If the switched linear system of (2.1) is quadratically stabilizable, then the switching rule is given by

$$\sigma(x, t) \stackrel{\triangle}{=} \underset{1\leq k\leq \mathcal{S}}{\mathrm{argmin}} \left\{x^T(t)\left(A_k^T\mathcal{P} + \mathcal{P}A_k\right)x(t)\right\}. \tag{2.4}$$

Theorem 2. [10] Assume $\mathcal{S} = 2$. The switched linear system of (2.1) is quadratically stabilizable via state feedback switching if and only if there exists nonnegative constants τ_1 and τ_2 satisfying $\tau_1 + \tau_2 = 1$ such that $\tau_1 A_1 + \tau_2 A_2$ is asymptotically stable, i.e. there exits the symmetric positive definite matrix $\mathcal{P}$ satisfying the inequality of (2.3).

If the switched linear system of (2.1) is quadratically stabilizable, then the switching rule is given by (2.4).

Lemma 1. For arbitrary vectors λ and ξ and matrices Λ and Ξ which have appropriate dimensions, the following relation holds.

$$\begin{aligned} H_e\left\{\lambda^T \Lambda^T \Delta(t) \Xi \xi\right\} &\leq 2\left\|\Lambda\lambda\right\|\left\|\Delta(t)\right\|\left\|\Xi\xi\right\| \\ &\leq 2\left\|\Lambda\lambda\right\|\left\|\Xi^T\xi\right\| \end{aligned} \tag{2.5}$$

where the time-varying matrix $\Delta(t)$ with an appropriate dimension satisfies the relation $\left\|\Delta(t)\right\| \leq 1.0$.

Lemma 2. (Schur complement) For a given constant real symmetric matrix Ξ, the following items are equivalent.

(i). $\Xi = \begin{pmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{12}^T & \Xi_{22} \end{pmatrix} > 0$

(ii). $\Xi_{11} > 0$ and $\Xi_{22} - \Xi_{12}^T \Xi_{11}^{-1} \Xi_{12} > 0$

(iii). $\Xi_{22} > 0$ and $\Xi_{11} - \Xi_{12} \Xi_{22}^{-1} \Xi_{12}^T > 0$.

3 Problem Formulation

Consider the following uncertain switched linear system.

$$\begin{aligned} \frac{d}{dt}x(t) &= A_{\sigma(x,t)}\left(\Delta, t\right)x(t) + B_{\sigma(x,t)}u(t) + \Upsilon_1\omega(t) \\ z(t) &= Cx(t) + \Upsilon_2\omega(t) \end{aligned} \tag{3.1}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $z(t) \in \Re^l$ and $\omega(t) \in \Re^r$ are the vectors of the state (assumed to be available for feedback), the control input, the controlled output and the disturbance input, respectively. The disturbance input $\omega(t)$ is assumed to be square integrable, i.e. $w(t) \in \mathcal{L}_2[0, \infty)$, and C and Υ_k $(k = 1, 2)$ in (3.1) denote the known constant matrices with appropriate dimensions respectively. Furthermore, $\sigma(x,t) \in \mathcal{I}_{\mathcal{S}} \stackrel{\triangle}{=} \{1, 2, \cdots, \mathcal{S}\}$ is a switching rule to be designed and the matrix $A_{\sigma(x,t)}(\Delta, t)$ is supposed to have an appropriate dimension and the following time-varying structure.

$$A_{\sigma(x,t)}\left(\Delta, t\right) = A_{\sigma(x,t)} + B_{\sigma(x,t)} D_{\sigma(x,t)} \Delta_{\sigma(x,t)}(t) \mathcal{E}_{\sigma(x,t)}. \tag{3.2}$$

The structure of the time-varying parameter which is shown in (3.2) is called “the uncertainties satisfying matching condition”. Note that the assumption that the matching condition holds is not uncommon and adopted in many literatures (e.g. [16, 19] and references therein). In (3.2), $A_{\sigma(x,t)}$ and $B_{\sigma(x,t)}$ denote the nominal values for the system parameters, and the matrices $D_{\sigma(x,t)} \in \Re^{m\times p}$ and $\mathcal{E}_{\sigma(x,t)} \in \Re^{q\times n}$ represent the structure of the uncertainties for the system matrix. The time-varying parameter $\Delta_{\sigma(x,t)}(t)$ represents unknown parameters with an appropriate dimension and satisfies the relation $\left\|\Delta_{\sigma(x,t)}(t)\right\| \leq 1.0$.

The nominal system, ignoring the unknown parameter $\Delta_{\sigma(x,t)}(t) \in \Re^{p\times q}$ in (3.1), is given by

$$\begin{aligned} \frac{d}{dt}\overline{x}(t) &= A_{\sigma(\overline{x},t)}\overline{x}(t) + B_{\sigma(\overline{x},t)}\overline{u}(t) + \Upsilon_1\omega(t) \\ \overline{z}(t) &= C\overline{x}(t) + \Upsilon_2\omega(t) \end{aligned} \tag{3.3}$$

where $\overline{x}(t) \in \Re^n$, $\overline{u}(t) \in \Re^m$ and $\overline{z}(t) \in \Re^l$ are the vectors of the state, the control input and the controlled output for the nominal system respectively.

Now we consider the state feedback law for the nominal system expressed as

$$\overline{u}(t) \stackrel{\triangle}{=} K_{\sigma(\overline{x},t)}\overline{x}(t) \tag{3.4}$$

where K_k $(k = 1, \cdots, \mathcal{S})$ is the fixed feedback gain matrix for the k-th subsystem and decision methods of the feedback gain matrix $K_k \in \Re^{m\times n}$ $(k = 1, \cdots, \mathcal{S})$ and the switching rule $\sigma(\overline{x}, t)$ will be stated in Section 4.

From (3.3) and (3.4), we have the following closed-loop system for the nominal system.

$$\begin{aligned} \frac{d}{dt}\overline{x}(t) &= A_{K_{\sigma(\overline{x},t)}}\overline{x}(t) + \varUpsilon_1\omega(t) \\ zw\overline{z}(t) &= C\overline{x}(t) + \varUpsilon_2\omega(t). \end{aligned} \tag{3.5}$$

In (3.5), $A_{K_{\sigma(\overline{x},t)}}$ is the matrix given by

$$A_{K_{\sigma(\overline{x},t)}} = A_{\sigma(\overline{x},t)} + B_{\sigma(\overline{x},t)}K_{\sigma(\overline{x},t)}. \tag{3.6}$$

For the uncertain switched linear system of (3.1), by using the fixed gain matrices K_k we consider the following control law.

$$u(t) \stackrel{\triangle}{=} K_{\sigma(x,t)}x(t) + \xi_{\sigma(x,t)}(\mathcal{L}, x, t) \tag{3.7}$$

where $\xi_{\sigma(x,t)}(x, \mathcal{L}, t) \in \Re^m$ is a compensation input[4] to correct the effect of uncertainties, and it is supposed to have the following structure.

$$\xi_{\sigma(x,t)}(\mathcal{L}, x, t) \stackrel{\triangle}{=} \mathcal{L}_{\sigma(x,t)}(x, t)x(t) \tag{3.8}$$

where $\mathcal{L}_{\sigma(x,t)}(x, t) \in \Re^{m\times n}$ is a variable gain matrix. Thus from (3.1), (3.2), (3.7) and (3.8), the closed-loop system for the uncertain switched linear system of (3.1) can be written as

$$\begin{aligned} \frac{d}{dt}x(t) &= \left(A_{K_{\sigma(x,t)}} + B_{\sigma(x,t)}D_{\sigma(x,t)}\Delta_{\sigma(x,t)}(t)\mathcal{E}_{\sigma(x,t)}\right)x(t) + B_{\sigma(x,t)}\mathcal{L}_{\sigma(x,t)}(x, t)x(t) + \varUpsilon_1\omega(t) \\ z(t) &= Cx(t) + \varUpsilon_2\omega(t). \end{aligned} \tag{3.9}$$

In (3.9), $A_{K_{\sigma(x,t)}}$ is a matrix given by $A_{K_{\sigma(x,t)}} = A_{\sigma(x,t)} + B_{\sigma(x,t)}K_{\sigma(x,t)}$.

Now on the basis of the existing results (e.g. [7, 8]) we shall give the definition of the variable gain robust control with guaranteed $\mathcal{L}_2$ gain performance $\gamma^* > 0$ for the uncertain switched linear system of (3.1) and the control input of (3.7).

Definition 2. For the uncertain switched linear system of (3.1), the control input of (3.7) is said to be a variable gain robust control with guaranteed $\mathcal{L}_2$ gain performance level $\gamma^* > 0$ if the closed-loop system of (3.9) is robustly stable (internally stable) and $\mathcal{H}^\infty$-norm of the closed-loop system transfer function from the disturbance input $w(t)$ to the controlled output $z(t)$ is less than or equal to a positive constant γ^*.

Now by introducing a symmetric positive definite matrix $\mathcal{P} \in \Re^{n\times n}$, we consider a quadratic function $\mathcal{V}(x, t) = x^T(t)\mathcal{P}x(t)$. Besides, we define the Halmiltonian defined as

$$\mathcal{H}(x, \omega, t) \stackrel{\triangle}{=} \frac{d}{dt}\mathcal{V}(x, t) + z^T(t)z(t) - (\gamma^*)^2 w^T(t)w(t). \tag{3.10}$$

Then we have the following lemma for a variable gain robust control with guaranteed $\mathcal{L}_2$ gain performance γ^*.

[†] The positive constant γ^* is said to disturbance attenuation level.

Lemma 3. Consider the uncertain switched linear system of (3.1) and the control input of (3.7).

For the quadratic function $\mathcal{V}(x,t) = x^T(t)\mathcal{P}x(t)$ and the signals $z(t)$ and $\omega(t)$, if there exist the symmetric positive definite matrix $\mathcal{P} \in \Re^{n\times n}$ and a positive scalar γ^* which satisfy

$$\mathcal{H}(x,\omega,t) < 0 \tag{3.11}$$

then the control input of (3.7) is a variable gain robust control with guaranteed $\mathcal{L}_2$ gain performance $\gamma^* > 0^\dagger$.

Proof. By integrating both sides of the inequality of (3.11) from 0 to ∞ with $x(0) = 0$, we easily see from $\mathcal{V}(x,0) = 0$ that the following inequality holds.

$$\int_0^\infty z^T(t)z(t)dt - (\gamma^*)^2\int_0^\infty w^T(t)w(t)dt + \mathcal{V}(x,\infty) < 0. \tag{3.12}$$

We see from the inequality of (3.12) that the closed-loop system of (3.9) is robustly stable (internally stable)[‡] and that the $\mathcal{H}^\infty$-norm of the closed-loop system transfer function from the disturbance input $w(t) \in \Re^q$ to the controlled output $z(t) \in \Re^p$ is less than or equal to a given positive constant γ^*, because the inequality of (3.12) means the following relation.

$$||z(t)||_{\mathcal{L}_2} < \gamma^*||w(t)||_{\mathcal{L}_2} \tag{3.13}$$

Thus the proof of **Lemma 3** is completed. □

Therefore, our control objective is to design the variable gain robust controller with guaranteed $\mathcal{L}_2$ gain performance γ^* for the uncertain switched linear system of (3.1). That is to derive the symmetric positive definite matrix $\mathcal{P} \in \Re^{n\times n}$, a positive scalar γ^*, the feedback gain matrices K_k $(k = 1,\cdots,\mathcal{S})$, the switching rule $\sigma(x,t)$ and the variable gain matrix $\mathcal{L}_{\sigma(x,t)}(x,t) \in \Re^{m\times l}$ which satisfy the inequality condition of (3.11) for all admissible uncertainties $\Delta_{\sigma(x,t)}(t) \in \Re^{p\times q}$ and the disturbance input $\omega(t) \in \mathcal{L}_2[0,\infty)$. Next section, a design method of the feedback gain matrices K_k $(k = 1,\cdots,\mathcal{S})$, the switching rule $\sigma(x,t)$, and the variable gain matrix $\mathcal{L}_{\sigma(x,t)}(x,t) \in \Re^{m\times l}$ is derived.

4 The Variable Gain Robust Controllers

In this section firstly, we show a design method of the switching rule $\sigma(x,t) \in \Re^1$ and the feedback gain matrices K_k $(k = 1,\cdots,\mathcal{S}) \in \Re^{m\times n}$ and next, the variable gain matrix $\mathcal{L}_{\sigma(x,t)}(x,t) \in \Re^{m\times n}$ is derived. The switching rule $\sigma(x,t)$ and the feedback gain matrices K_k are determined by using the nominal closed-loop system of (3.5), and the variable gain matrix $\mathcal{L}_{\sigma(x,t)}(x,t) \in \Re^{m\times n}$ is derived such that the effect of uncertainties is reduced.

Firstly using the nominal closed-loop system of (3.5), we show the following lemma for the switching rule $\sigma(\overline{x},t)$ and the feedback gain matrices K_k $(k = 1,\cdots,\mathcal{S})$.

Lemma 4. Consider the switched linear system of (3.3) with the control input of (3.4).

The switched linear system of (3.5) is robustly stable (internally stable) with guaranteed $\mathcal{L}_2$ gain performance $\gamma^* > 0$ if there exist constant scalars $\tau_1 \geq 0,\cdots,\tau_{\mathcal{S}} \geq 0$ with $\sum_{k=1}^{\mathcal{S}}\tau_k > 0$ (if $\mathcal{S} = 2$

[‡] Note that if the condition of (3.11) is satisfied, then we see from the relation $\omega(t) \equiv 0$, the **Definition 1** and **Theorem 1** that the quadratic function $\mathcal{V}(x,t)$ becomes a Lyapunov function for the closed-loop system of (3.9). Namely, quadratic stability (internal stability) is guaranteed for the closed-loop system of (3.9).

then $\tau_1 + \tau_2 = 1$), a symmetric positive definite matrix $\mathcal{X} \stackrel{\triangle}{=} \mathcal{P}^{-1} \in \Re^{n\times n}$, the matrix $\mathcal{W}_k \in \Re^{m\times n}$ $(k = 1, \cdots, \mathcal{S})$ and a positive scalar γ which satisfy the inequality condition

$$\left(\begin{array}{c|c|c} H_e\left\{\left(\sum_{k=1}^{\mathcal{S}}\tau_k A_k\right)\mathcal{X} + \sum_{k=1}^{\mathcal{S}}\tau_k B_k \mathcal{W}_k\right\} & \varUpsilon_1 + \mathcal{X}C^T\varUpsilon_2 & \mathcal{X}C^T \\ \hline \star & \varUpsilon_2^T\varUpsilon_2 - \gamma I_p & 0 \\ \hline \star & \star & -I_l \end{array}\right) < 0. \tag{4.1}$$

then the switched linear system of (3.5) is robustly stable via the control input of (3.7) with the feedback gain matrices K_k $(k = 1, \cdots, \mathcal{S})$ given by

$$K_k = \mathcal{W}_k \mathcal{X}^{-1} \tag{4.2}$$

and the switching rule which is taken as

$$\sigma(\overline{x}, t) \stackrel{\triangle}{=} \underset{1\le k\le \mathcal{S}}{\mathrm{argmin}} \left\{\overline{x}^T(t)\left(A_{K_k}^T\mathcal{P} + \mathcal{P}A_{K_k}\right)\overline{x}(t)\right\}. \tag{4.3}$$

Besides, the disturbance attenuation level $\gamma^* > 0$ is given by $\gamma^* = \sqrt{\gamma}$.

Proof. Now, we consider the inequality condition of (3.11) for the switched linear system of (3.5), i.e. $\mathcal{H}(\overline{x}, \omega, t) < 0$. Then the result of **Lemma 4** is straightforward from **Theorem 1**, Proof of **Lemma 4** and the existing results (e.g. [12, 13, 14]). □

Next, we state and prove the following theorem that is main results in this paper for the proposed control with guaranteed $\mathcal{L}_2$ gain performance.

Theorem 3. Consider the uncertain switched linear system of (3.1) and the control input of (3.7).

For the uncertain switched linear system of (3.1) with the control input of (3.7) if there exist the symmetric positive definite matrix $\mathcal{X} \in \Re^{n\times n}$ and the matrices $\mathcal{W}_k$ $(k = 1, \cdots, \mathcal{S})$ which satisfy the matrix inequality condition of (4.1), then using the symmetric positive definite matrix $\mathcal{P} \stackrel{\triangle}{=} \mathcal{X}^{-1}$, we consider the following variable gain matrix $\mathcal{L}_{\sigma(x,t)}(x, t) \in \Re^{m\times n}$.

$$\mathcal{L}_{\sigma(x,t)}(x,t) = \begin{cases} -\dfrac{\left\|\mathcal{D}_{\sigma(x,t)}^T B_{\sigma(x,t)}^T\mathcal{P}x(t)\right\|\left\|\mathcal{E}_{\sigma(x,t)}x(t)\right\|}{\left\|B_{\sigma(x,t)}^T\mathcal{P}x(t)\right\|^2} B_{\sigma(x,t)}^T\mathcal{P} & \text{if} \quad B_{\sigma(x,t)}^T\mathcal{P}x(t) \neq 0 \\ \mathcal{L}_{\sigma(x,t_\zeta)}(x,t_\zeta) & \text{if} \quad B_{\sigma(x,t)}^T\mathcal{P}x(t) = 0 \end{cases} \tag{4.4}$$

where $t_\zeta = \lim_{\zeta>0,\zeta\to 0}(t-\zeta)$[3]. Furthermore using the matrices $\mathcal{X} \in \Re^{n\times n}$ and $\mathcal{W}_k \in \Re^{m\times n}$ $(k = 1, \cdots, \mathcal{S})$ which satisfy the matrix inequality condition of (4.1), the feedback gain matrices K_k $(k = 1, \cdots, \mathcal{S}) \in \Re^{m\times n}$ are derived as (4.2). Additionally, the switching rule $\sigma(x, t)$ in the uncertain closed-loop system of (3.9) is given by

$$\sigma(x, t) \stackrel{\triangle}{=} \underset{1\le k\le \mathcal{S}}{\mathrm{argmin}} \left\{x^T(t)\left(A_{K_k}^T\mathcal{P} + \mathcal{P}A_{K_k}\right)x(t)\right\}. \tag{4.5}$$

Then the control input of (3.7) is a variable gain robust control with guaranteed $\mathcal{L}_2$ gain performance $\gamma^* > 0$.

Proof. Using the symmetric positive definite matrix $\mathcal{P} = \mathcal{X}^{-1}$ where $\mathcal{X}$ is the symmetric positive definite matrix satisfying the matrix inequality of (4.1), we consider the condition of (3.11). The time derivative of the quadratic function $\mathcal{V}(x) = x^T(t)\mathcal{P}x(t)$ along the trajectory of the uncertain closed-loop system of (3.9) can be written as

$$\begin{aligned}\frac{d}{dt}\mathcal{V}(x) = x^T(t)\left[H_e\left\{\mathcal{P}A_{K_{\sigma(x,t)}}\right\}\right]x(t) + 2x^T(t)\mathcal{P}B_{\sigma(x,t)}D_{\sigma(x,t)}\Delta_{\sigma(x,t)}(t)\mathcal{E}_{\sigma(x,t)}x(t) \\ + 2x^T(t)\mathcal{P}B_{\sigma(x,t)}\mathcal{L}_{\sigma(x,t)}(x,t)x(t) + 2x^T(t)\mathcal{P}\varUpsilon_1\omega(t).\end{aligned} \tag{4.6}$$

Thus we see from the definition of Hamiltonian $\mathcal{H}(x,\omega,t)$ that the following relation is satisfied.

$$\mathcal{H}(x,\omega,t) = \begin{pmatrix} x(t) \\ w(t) \end{pmatrix} \left(\begin{array}{c:c} H_e\left\{\mathcal{P}A_{K_{\sigma(x,t)}}\right\} + C^TC & \mathcal{P}\Upsilon_1 + C^T\Upsilon_2 \\ \hdashline \star & \Upsilon_2^T\Upsilon_2 - (\gamma^*)^2 I_r \end{array}\right) \begin{pmatrix} x(t) \\ w(t) \end{pmatrix}$$
$$+ 2x^T(t)\mathcal{P}B_{\sigma(x,t)}D_{\sigma(x,t)}\Delta_{\sigma(x,t)}(t)\mathcal{E}_{\sigma(x,t)}x(t) + 2x^T(t)\mathcal{P}B_{\sigma(x,t)}\mathcal{L}_{\sigma(x,t)}(x,t)x(t). \quad (4.7)$$

Applying **Lemma 1** to the second term of the right hand side in (4.7) we obtain

$$\mathcal{H}(x,\omega,t) \leq \begin{pmatrix} x(t) \\ w(t) \end{pmatrix} \left(\begin{array}{c:c} H_e\left\{\mathcal{P}A_{K_{\sigma(x,t)}}\right\} + C^TC & \mathcal{P}\Upsilon_1 + C^T\Upsilon_2 \\ \hdashline \star & \Upsilon_2^T\Upsilon_2 - (\gamma^*)^2 I_r \end{array}\right) \begin{pmatrix} x(t) \\ w(t) \end{pmatrix}$$
$$+ 2\left\|D^T_{\sigma(x,t)}B^T_{\sigma(x,t)}\mathcal{P}x(t)\right\|\left\|\mathcal{E}_{\sigma(x,t)}x(t)\right\| + 2x^T(t)\mathcal{P}B_{\sigma(x,t)}\mathcal{L}_{\sigma(x,t)}(x,t)x(t). \quad (4.8)$$

Besides from the existing works of Wicks et al.[9] and Oya and Hagino[15], combining the condition of (2.3) and the switching rule $\sigma(x,t)$ expressed as (4.5), we obtain the following inequality for internal stability (quadratic stablity).

$$\sum_{k=1}^{\mathcal{S}}\tau_k x^T(t)\left(A^T_{K_{\sigma(x,t)}}\mathcal{P} + \mathcal{P}A_{K_{\sigma(x,t)}}\right)x(t) \leq x^T(t)\left\{\left(\sum_{k=1}^{\mathcal{S}}\tau_k A_{K_k}\right)^T\mathcal{P} + \mathcal{P}\left(\sum_{k=1}^{\mathcal{S}}\tau_k A_{K_k}\right)\right\}x(t). \quad (4.9)$$

Therefore, one can easily see from the inequalities of (4.8), the condition of (4.9) and the relation $\gamma = (\gamma^*)^2$ that the following relation holds.

$$\mathcal{H}(x,\omega,t) \leq \begin{pmatrix} x(t) \\ w(t) \end{pmatrix} \left(\begin{array}{c:c} H_e\left\{\left(\sum_{k=1}^{\mathcal{S}}\tau_k A_{K_k}\right)^T\mathcal{P}\right\} + C^TC & \mathcal{P}\Upsilon_1 + C^T\Upsilon_2 \\ \hdashline \Upsilon_1^T\mathcal{P} + \Upsilon_2^TC & \Upsilon_2^T\Upsilon_2 - \gamma I_r \end{array}\right) \begin{pmatrix} x(t) \\ w(t) \end{pmatrix}$$
$$+ 2\left\|D^T_{\sigma(x,t)}B^T_{\sigma(x,t)}\mathcal{P}x(t)\right\|\left\|\mathcal{E}_{\sigma(x,t)}x(t)\right\| + 2x^T(t)\mathcal{P}B_{\sigma(x,t)}\mathcal{L}_{\sigma(x,t)}(x,t)x(t). \quad (4.10)$$

Now we consider the case of $B^T_{\sigma(x,t)}\mathcal{P}x(t) \neq 0$. From the variable gain matrix of (4.4), we have

$$\mathcal{H}(x,\omega,t) \leq \begin{pmatrix} x(t) \\ w(t) \end{pmatrix} \left(\begin{array}{c:c} H_e\left\{\left(\sum_{k=1}^{\mathcal{S}}\tau_k A_{K_k}\right)^T\mathcal{P}\right\} + C^TC & \mathcal{P}\Upsilon_1 + C^T\Upsilon_2 \\ \hdashline \star & \Upsilon_2^T\Upsilon_2 - \gamma I_r \end{array}\right) \begin{pmatrix} x(t) \\ w(t) \end{pmatrix} \quad (4.11)$$

and thus we consider the following condition instead of (3.11).

$$\left(\begin{array}{c:c} H_e\left\{\left(\sum_{k=1}^{\mathcal{S}}\tau_k A_{K_k}\right)^T\mathcal{P}\right\} + C^TC & \mathcal{P}\Upsilon_1 + C^T\Upsilon_2 \\ \hdashline \star & \Upsilon_2^T\Upsilon_2 - \gamma I_r \end{array}\right) < 0. \quad (4.12)$$

By introducing a complementary variable $\mathcal{W}_k = \mathcal{K}_k\mathcal{X}$ and pre- and post-multiplying (4.12) by $\mathrm{diag}(\mathcal{X}, I_r)$, we have

$$\left(\begin{array}{c:c} H_e\left\{\left(\sum_{k=1}^{\mathcal{S}}\tau_k A_k\right)\mathcal{X} + \sum_{k=1}^{\mathcal{S}}\tau_k B_k\mathcal{W}_k\right\} + \mathcal{X}C^TC\mathcal{X} & \Upsilon_1 + \mathcal{X}C^T\Upsilon_2 \\ \hdashline \star & \Upsilon_2^T\Upsilon_2 - \gamma I_r \end{array}\right) < 0. \quad (4.13)$$

Note that the left hand side of the matrix inequality of (4.13) can be rewritten as

$$\begin{aligned}&\left(\begin{array}{c|c} H_e\left\{\left(\sum_{k=1}^{\mathcal{S}}\tau_k A_k\right)\mathcal{X}+\sum_{k=1}^{\mathcal{S}}\tau_k B_k\mathcal{W}_k\right\}+\mathcal{X}C^TC\mathcal{X} & \Upsilon_1+\mathcal{X}C^T\Upsilon_2 \\ \hline \star & \Upsilon_2^T\Upsilon_2-\gamma I_r\end{array}\right)\\ &=\left(\begin{array}{c|c} H_e\left\{\left(\sum_{k=1}^{\mathcal{S}}\tau_k A_k\right)\mathcal{X}+\sum_{k=1}^{\mathcal{S}}\tau_k B_k\mathcal{W}_k\right\} & \Upsilon_1+\mathcal{X}C^T\Upsilon_2 \\ \hline \star & \Upsilon_2^T\Upsilon_2-\gamma I_r\end{array}\right)-\left(\begin{array}{c}\mathcal{X}C^T\\ \hline 0\end{array}\right)(-I_l)\left(\begin{array}{c|c}C\mathcal{X} & 0\end{array}\right).\end{aligned} \quad (4.14)$$

Thus applying **Lemma 2** (Schur complement) to (4.13), we find that the inequality of (4.13) is equivalent to the LMI

$$\left(\begin{array}{c|c|c} H_e\left\{\left(\sum_{k=1}^{\mathcal{S}}\tau_k A_k\right)\mathcal{X}+\sum_{k=1}^{\mathcal{S}}\tau_k B_k\mathcal{W}_k\right\} & \Upsilon_1+\mathcal{X}C^T\Upsilon_2 & \mathcal{X}C^T \\ \hline \star & \Upsilon_2^T\Upsilon_2-\gamma I_p & 0 \\ \hline \star & \star & -I_l\end{array}\right)<0 \quad (4.15)$$

and one can easily see that if the matrix inequality of (4.1) holds then the condition of (3.11) is also satisfied.

Next, we consider the case of $B^T_{\sigma(x,t)}\mathcal{P}x(t)=0$. In this case, we easily see from (4.7) that the Hamiltonian $\mathcal{H}(x,\omega,t)$ satisfies the relation of (3.11), provided that if there exists the symmetric positive definite matrix $\mathcal{X}\in\Re^{n\times n}$ and positive scalars τ_k which satisfy the inequality of (4.1). In this case, we assume that the variable gain matrix $\mathcal{L}_{\sigma(x,t)}(x,t)\in\Re^{m\times n}$ is defined as $\mathcal{L}_{\sigma(x,t)}(x,t)=\mathcal{L}_{\sigma(x,t)}(x,t_\zeta)$ where $t_\zeta=\lim_{\zeta>0,\zeta\to0}(t-\zeta)$[3] .

Obviously from the above discussion, the control input of (3.7) with the feedback gain matrices $K_k\in\mathbb{R}^{m\times n}$ of (4.2), the switching rule $\sigma(x,t)$ of (4.3) and the variable gain matrix $\mathcal{L}_{\sigma(x,t)}(x,t)$ of (4.4) is a variable gain robust control with guaranteed $\mathcal{L}_2$ gain performance $\gamma^*>0$. Therefore the proof of **Theorem 3** is accomplished. □

Remark 1. If the positive scalars τ_k $(k=1,\cdots,\mathcal{S})$ are fixed, then the matrix inequality condition of (4.1) becomes a linear matrix inequality (LMI) in $\mathcal{X}$, W_k and γ and thus the LMI of (4.1) defines a convex solutionset of $(\mathcal{X},\mathcal{W}_k,\gamma)$. Thus various efficient convex optimization algolithms can be utilized to test whether the LMI is solvable and to generate particular solutions. Namely, we can derive the variable gain robust controller minimizing the disturbance attenuation level γ^* (see **Appendix A.1** for details).

Remark 2. In this paper, the fixed gain matrices K_k $(k=1,\cdots,\mathcal{S})$ are determined by using the nominal system. The problem needed to solved in the proposed design is to find constant scalars $\tau_k\geq0$ $(k=1,\cdots,\mathcal{S})$ with $\sum_{k=1}^{\mathcal{S}}\tau_k>0$, a positive scalar γ^*, a symmetric positive definite matrix $\mathcal{X}\in\Re^{n\times n}$ and matrices $\mathcal{W}_k\in\Re^{m\times n}$ $(k=1,\cdots,\mathcal{S})$. If for the uncertain switched linear system of (3.1), the number of subsystems is $\mathcal{S}=\mathcal{M}$, then the number of the inequalities needed to solved in the proposed control design is "1" and that in the existing results is "$2^{\mathcal{M}}$" (see [15]). Namely, there are always less matrix inequalities needed to be solved in proposed design method than that in the existing result (see [11, 12, 13]). In other word, the proposed variable gain robust controller is less conservative. Thus the proposed approach is useful.

Remark 3. The uncertainties in the switched linear system of (3.1) are included in the state matrix only. The proposed design method can also be applied to the case that the parameter uncertainty is contained in both the system matrix and the input matrix. By introducing additional actuator dynamics and constituting an augmented system, uncertainties in the input matrix are embedded in the system matrix of the augmented system[19]. Therefore the same design procedure can be applied. Besides although the uncertainty in the controlled system under consideration is defined as norm-bounded uncertainties and satisfies the matching condition, we can easily extend the result in this paper to some uncertain switched linear systems such as switched linear systems with structured uncertainties or mismatched one (see [15]).

5 Illustrative Examples

In order to demonstrate the efficiency of the proposed robust controller, we have run a simple example. The control problem considered here are not necessary practical. However, the simulation results stated below illustrate the distinct feature of the proposed variable gain robust controller with guaranteed $\mathcal{L}_2$ gain performance.

Consider the three-dimesional uncertain switched linear system with the following coefficient matrices.

$$\begin{aligned}
&A_1 = \begin{pmatrix} -1.0 & 1.0 & -1.5 \\ 0.0 & 1.0\times 10^{-1} & 0.0 \\ -1.0 & -2.5 & 5.0\times 10^{-1} \end{pmatrix}, \; B_1 = \begin{pmatrix} 3.0 \\ 0.0 \\ 1.0 \end{pmatrix} \\
&A_2 = \begin{pmatrix} 1.0\times 10^{-1} & 0.0 & 0.0 \\ -0.5 & -1.0 & -1.5 \\ 8.0\times 10^{-1} & 1.0 & -2.0 \end{pmatrix}, \; B_2 = \begin{pmatrix} 0.0 \\ 2.0 \\ 1.0 \end{pmatrix} \\
&C = \begin{pmatrix} 1.0 & 1.0 & 0.0 \end{pmatrix}, \; \Upsilon_1 = \begin{pmatrix} 1.0 & 0.0 & 1.0 \end{pmatrix}^T, \; \Upsilon_2 = 1.0 \\
&\mathcal{D}_1 = \begin{pmatrix} 3.0 & 5.0\times 10^{-1} \end{pmatrix}, \; \mathcal{E}_1 = \begin{pmatrix} 0.0 & 1.0 & 0.0 \\ 0.0 & 1.0 & 1.0 \end{pmatrix}, \\
&\mathcal{D}_2 = \begin{pmatrix} 0.0 & -5.0 \end{pmatrix}, \; \mathcal{E}_2 = \begin{pmatrix} 1.0 & 1.0 & 1.0 \\ 1.0 & 0.0 & 0.0 \end{pmatrix}.
\end{aligned} \tag{5.1}$$

Namely $\sigma(x,t) \in \mathcal{I}_2 \overset{\triangle}{=} \{1,2\}$, i.e. $\mathcal{S} = 2$. Note that each nominal subsystem for the uncertain switched linear system with coefficient matrices of (5.1) is not controllable.

Now by selecting the parameters τ_1 and τ_2 such as $\tau_1 = \tau_2 = 0.5$ (see **Theorem 2**) and applying **Theorem 3**, i.e. solving the matrix inequality of (4.1), we obtain

$$\begin{aligned}
&\mathcal{X} = \begin{pmatrix} 3.10989\times 10^{1} & -6.44230 & 3.39236 \\ \star & 2.07461\times 10^{1} & 1.11133\times 10^{1} \\ \star & \star & 3.51995\times 10^{1} \end{pmatrix}, \\
&\mathcal{W}_1 = \begin{pmatrix} -2.74944\times 10^{4} & 0.0 & -9.16700\times 10^{3} \end{pmatrix}, \\
&\mathcal{W}_2 = \begin{pmatrix} 1.27972 & -4.67610 & -1.91051 \end{pmatrix}\times 10^{4}, \\
&\gamma = 8.09887\times 10^{-1}.
\end{aligned} \tag{5.2}$$

and then the feed back gain matrices $K_1 \in \Re^{1\times 3}$ and $K_2 \in \Re^{1\times 3}$ can be derived as

$$\begin{aligned}
K_1 &= \begin{pmatrix} -9.21870\times 10^{2} & -2.33861\times 10^{2} - 9.78344\times 10^{1} \end{pmatrix} \\
K_2 &= \begin{pmatrix} -1.13259\times 10^{2} & -2.41220\times 10^{3} & 2.29738\times 10^{2} \end{pmatrix}.
\end{aligned} \tag{5.3}$$

Besides from the definition of γ, the guaranteed $\mathcal{L}_2$ gain performance $\gamma^* > 0$ for the uncertain switched linear system with coefficient matrices of (5.1) is given by

$$\gamma^* \overset{\triangle}{=} \sqrt{\gamma} = 8.09887\times 10^{-1}. \tag{5.4}$$

Thus, the proposed variable gain robust controller with guaranteed $\mathcal{L}_2$ gain performance can be obtained by solving a matrix inequality of (4.1) only and the variable gain matrix for $B^T_{\sigma(x,t)}\mathcal{P}x(t) \neq 0$

can be expressed as

$$\mathcal{L}_{\sigma(x,t)}(x,t) = \begin{cases} -\dfrac{\left\|\mathcal{D}^T_{\sigma(x,t)}B^T_{\sigma(x,t)}\mathcal{P}x(t)\right\| \left\|\mathcal{E}_{\sigma(x,t)}x(t)\right\|}{\left\|\begin{pmatrix} 1.00592\times 10^1 & 2.55297\times 10^2 & 1.06545\times 10^2 \end{pmatrix} x(t)\right\|^2} B^T_{\sigma(x,t)}\mathcal{P} \\ \quad \text{if} \quad \sigma(x,t) = 1 \\ -\dfrac{\left\|\mathcal{D}^T_{\sigma(x,t)}B^T_{\sigma(x,t)}\mathcal{P}x(t)\right\| \left\|\mathcal{E}_{\sigma(x,t)}x(t)\right\|}{\left\|\begin{pmatrix} 2.31937\times 10^2 & 1.07820\times 10^1 & -7.86726\times 10^3 \end{pmatrix} x(t)\right\|^2} B^T_{\sigma(x,t)}\mathcal{P} \\ \quad \text{if} \quad \sigma(x,t) = 2 \end{cases} \tag{5.5}$$

where $\mathcal{D}^T_{\sigma}(x,t)B^T_{\sigma}(x,t)\mathcal{P}x(t)$ and $\mathcal{E}_{\sigma(x,t)}x(t)$ for the switched signal $\sigma(x,t)$ can be computed as

$$\begin{aligned} &\bullet\ \sigma(x,t) = 1 \\ &\mathcal{D}^T_1 B^T_1 \mathcal{P}x(t) = \begin{pmatrix} 3.01778\times 10^1 & 7.65892\times 10^2 & 3.19635\times 10^2 \\ 5.02961\times 10^2 & 1.27647\times 10^2 & 5.32725\times 10^3 \end{pmatrix} x(t), \\ &\mathcal{E}_1 x(t) = \begin{pmatrix} 0.0 & 1.0 & 0.0 \\ 0.0 & 1.0 & 1.0 \end{pmatrix} x(t) \\ &\bullet\ \sigma(x,t) = 2 \\ &\mathcal{D}^T_2 B^T_2 \mathcal{P}x(t) = \begin{pmatrix} 0.0 & 0.0 & 0.0 \\ -1.15969\times 10^1 & -5.39101\times 10^1 & 3.93363\times 10^2 \end{pmatrix} x(t), \\ &\mathcal{E}_2 x(t) = \begin{pmatrix} 1.0 & 1.0 & 1.0 \\ 1.0 & 0.0 & 0.0 \end{pmatrix} x(t). \end{aligned} \tag{5.6}$$

On the other hand for the fixed gain state feedback control with $\mathcal{L}_2$ gain performance, the LMIs of (A.7) are not feasible. Therefore, we see that the fixed gain robust control with $\mathcal{L}_2$ gain performance for the uncertain switched linear systems with coefficient matrices of (5.1) can not be derived.

From the above, the effectiveness of the proposed design method has been presented.

6 Conclusions

In this paper, a design method of a variable gain robust controller with guaranteed $\mathcal{L}_2$ gain performance for a class of uncertain switched linear systems has been suggested, and a numerical example has presented to demonstrate the effectiveness of the proposed variable gain robust controller.

The proposed design scheme of the variable gain robust controller consists of designing the switching rule and the feedback gain matrices which are determined by using the nominal system and deriving the variable gain matrix in order to reduce the effect of uncertainties. In this paper, we have shown that the proposed variable gain robust controller with guaranteed $\mathcal{L}_2$ gain performance can be obtained by solving a matrix inequality condition which is same one for the nominal system. Therefore since the number of matrix inequalities needed to be solved is always less than the conventional fixed gain robust control based on the existing results (e.g. [11, 12, 13]), the proposed controller design approach is very useful. Furthermore, one can easily see that the result in this paper is an extension of the existing results such as [14], [15] and so on. Namely, the proposed design method can also be extended to switched linear systems with general structured/norm-bounded uncertainties.

The future research subjects are extensions of the proposed variable gain robust controller to such a broad class of systems as time-delay systems, output feedback control system and so on. Besides, we will also tackle the design problem of the variable gain robust controller for discrete-time switched systems and finite-time stabilization (e.g. [20]).

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Competing Interests

The authors declare that no competing interests exist.

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Appendix

In this Appendix, we discuss optimal $\mathcal{L}_2$ gain performance for the proposed variable gain robust controller and we show a design method of a fixed gain robust control with guaranteed $\mathcal{L}_2$ gain performance. The problem for optimal $\mathcal{L}_2$ gain performance can be reduced to convex constrained optimization problem and the fixed gain robust control with guaranteed $\mathcal{L}_2$ gain performance can be obtained by solving LMIs.

A.1 Optimal $\mathcal{L}_2$ gain performance

In this section, we consider the design problem of the proposed robust controller which minimizes the disturbance attenuation level γ^*. Since the matrix inequality of (4.1) is an LMI provided that the positive constants τ_k $(k = 1, \cdots, \mathcal{S})$ are fixed by designers, the problem of designing the proposed variable gain robust controller which minimizes the disturbance attenuation level γ^* is reduced to the following constrained optimization problem.

$$\underset{\mathcal{X}, \mathcal{W}_k, \gamma}{\mathrm{Minimize}}[\gamma] \quad \text{subject to } \ \mathcal{X} > 0, \ \ \gamma > 0 \ \text{ and } \ (4.1). \tag{A.1}$$

If the solution $\mathcal{X}^{\mathrm{opt}} > 0$, W_k^{opt} and $\gamma^{\mathrm{opt}} > 0$ of the constrained optimization problem of (A.1) is obtained, then the fixed feedback gain matrices K_k can be computed as (4.2) and the optimal value $\gamma^\star$ for the disturbance attenuation level γ^* is given by $\gamma^\star = \sqrt{\gamma^{\mathrm{opt}}}$. Note that, the constrained optimization problem of (4.15) can be solved by software such as MATLAB's LMI Control Toolbox, Scilab's LMITOOL and so on.

Consequently, the following theorem for designing the proposed variable gain robust controller with optimal disturbance attenuation level γ_{opt} can be developed.

Theorem A.1. Consider the uncertain switched linear system of (3.1) and the control input of (3.7).

If there exist the solution $\mathcal{X}^{\mathrm{opt}} > 0$, W_k^{opt} and $\gamma^{\mathrm{opt}} > 0$ of the constrained optimization problem of (A.1), then using the positive definite matrix $\mathcal{P}^{\mathrm{opt}} \stackrel{\triangle}{=} \left(\mathcal{X}^{\mathrm{opt}}\right)^{-1}$, we consider the following variable gain matrix $\mathcal{L}_{\sigma(x,t)}(x,t) \in \Re^{m\times n}$.

$$\mathcal{L}_{\sigma(x,t)}(x,t) = \begin{cases} -\dfrac{\left\|\mathcal{D}_\sigma^T(x,t)B_\sigma^T(x,t)\mathcal{P}^{\mathrm{opt}}x(t)\right\| \left\|\mathcal{E}_{\sigma(x,t)}x(t)\right\|}{\left\|B_{\sigma(x,t)}^T\mathcal{P}^{\mathrm{opt}}x(t)\right\|^2} B_{\sigma(x,t)}^T\mathcal{P}^{\mathrm{opt}} & \text{if} \quad B_{\sigma(x,t)}^T\mathcal{P}^{\mathrm{opt}}x(t) \neq 0 \\ \mathcal{L}_{\sigma(x,t_\zeta)}(x,t_\zeta) & \text{if} \quad B_{\sigma(x,t)}^T\mathcal{P}^{\mathrm{opt}}x(t) = 0 \end{cases} \tag{A.2}$$

where $t_\zeta = \lim_{\zeta>0,\zeta\to 0}(t-\zeta)$[3]. Furthermore the feedback gain matrices K_k $(k = 1, \cdots, \mathcal{S}) \in \Re^{m\times n}$ are derived as

$$K_k = \left(\mathcal{W}_k^{\mathrm{opt}}\right) \cdot \left(\mathcal{X}^{\mathrm{opt}}\right)^{-1} \tag{A.3}$$

Additionally, the switching rule $\sigma(x,t)$ in the uncertain closed-loop system of (3.9) is given by

$$\sigma(x,t) \stackrel{\triangle}{=} \underset{1\le k\le \mathcal{S}}{\mathrm{argmin}} \left\{ x^T(t) \left(A_{K_k}^T \mathcal{P}^{\mathrm{opt}} + \mathcal{P}^{\mathrm{opt}} A_{K_k} \right) x(t) \right\}. \tag{A.4}$$

Then the control input of (3.7) is a variable gain robust control with guaranteed $\mathcal{L}_2$ gain performance $\gamma^\star = \sqrt{\gamma^{\mathrm{opt}}}$.

A.2 $\mathcal{L}_2$ gain performance via a fixed gain state feedback control

This section shows the result for a fixed gain state feedback control with guaranteed $\mathcal{L}_2$ gain performance. Namely for the uncertain switched linear system of (3.1) we consider the following state feedback control with fixed gain matrices $F_{\sigma(x,t)}$.

$$u(t) \triangleq F_{\sigma(x,t)}x(t). \tag{A.5}$$

Thus one can see that the closed-loop system can be written as

$$\begin{aligned} \frac{d}{dt}x(t) &= \left(A_{F_{\sigma(x,t)}} + B_{\sigma(x,t)}D_{\sigma(x,t)}\Delta_{\sigma(x,t)}(t)\mathcal{E}_{\sigma(x,t)}\right)x(t) + \varUpsilon_1\omega(t) \\ zwz(t) &= Cx(t) + \varUpsilon_2\omega(t) \end{aligned} \tag{A.6}$$

where $A_{F_{\sigma(x,t)}}$ is a matrix given by $A_{F_{\sigma(x,t)}} = A_{\sigma(x,t)} + B_{\sigma(x,t)}F_{\sigma(x,t)}$. Then following theorem gives an LMI-based condition for the existance of a fixed gain state feedback control of (A.5) with guaranteed $\mathcal{L}_2$ gain performance.

Theorem A.2. Consider the uncertain switched linear system of (3.1) and the control input of (A.5).

For $l = 1, \cdots, \mathcal{S}$ if there exists symmetric positive definite matrix $\mathcal{Y}$, matrices $\mathcal{Z}_k$ $(k = 1, \cdots, \mathcal{S})$ and positive constants δ for LMIs

$$\left(\begin{array}{c|c|c|c} H_e\left\{\left(\sum_{k=1}^{\mathcal{S}}\tau_k A_k\right)\mathcal{Y} + \sum_{k=1}^{\mathcal{S}}\tau_k B_k\mathcal{Z}_k\right\} + \delta B_l\mathcal{D}_l\mathcal{D}_l^T B_l^T & \varUpsilon_1 + \mathcal{Y}C^T\varUpsilon_2 & \mathcal{Z}C^T & \mathcal{Z}\mathcal{E}_l^T \\ \hline \star & \varUpsilon_2^T\varUpsilon_2 - \gamma I_p & 0 & 0 \\ \hline \star & \star & -I_l & 0 \\ \hline \star & \star & \star & -\delta I_q \end{array}\right) < 0. \tag{A.7}$$

the switched linear system of (A.6) is robustly stable via the control input of (A.5) with the feedback gain matrices F_k $(k = 1, \cdots, \mathcal{S})$ given by

$$F_k = \mathcal{Z}_k\mathcal{Y}^{-1}. \tag{A.8}$$

Besides the switching rule $\sigma(x,t)$ is given by

$$\sigma(x,t) \triangleq \underset{1\le k\le\mathcal{S}}{\operatorname{argmin}}\left\{x^T(t)\left(A_{K_k}^T\mathcal{Y}^{-1} + \mathcal{Y}^{-1}A_{K_k}\right)x(t)\right\}. \tag{A.9}$$

Then the ucnertain closed-loop system of (A.6) is robustly stable (internally stable) with guaranteed $\mathcal{L}_2$ gain performance $\gamma^* = \sqrt{\gamma} > 0$.

Proof. One can see that the result of **Theorem A.2** can easily be proved as using the same procedure of the proof of **Lemma 4**. □

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