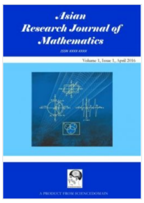

Asian Research Journal of Mathematics
1(1): 1-10, 2016, Article no.ARJOM.27387

SCIENCEDOMAIN *international*

www.sciencedomain.org

Generalization of Convolution Sums with Fibonacci Numbers and Lucas Numbers

Aeran Kim[1*]

[1] *A Private Mathematics Academy, 1599-6-Bungi, Keumam-2-dong, Deokjin-gu, Jeonju-si, Jeollabuk-do, 561-806, South Korea.*

Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/ARJOM/2016/27387

Editor(s):

(1) Sheng Zhang, School of Mathematics and Physics, Bohai University, Jinzhou, China.

Received: 31st May 2016

Accepted: 2nd July 2016

Published: 9th July 2016

Original Research Article

Abstract

In this article we construct some convolutions sums related to Fibonacci and Lucas numbers and obtain their generalized formulae.

Keywords: Fibonacci and Lucas numbers; convolution sums.

2010 Mathematics Subject Classification: 11B39.

1 Introduction

Let $\mathbb{N}$ be the set of positive integers. We may define the Fibonacci numbers, F_n, by

$$F_0 = 0, \qquad F_1 = 1, \qquad \text{and,} \qquad F_{n+2} = F_{n+1} + F_n.$$

Associated with the numbers of Fibonacci are the numbers of Lucas, L_n, which we may define by

$$L_0 = 2, \qquad L_1 = 1, \qquad \text{and,} \qquad L_{n+2} = L_{n+1} + L_n.$$

Fibonacci numbers and Lucas numbers can also be extended to negative index n satisfying

$$F_{-n} = (-1)^{n+1} F_n \qquad \text{and} \qquad L_{-n} = (-1)^n L_n. \tag{1.1}$$

*Corresponding author: E-mail: ae_ran_kim@hotmail.com

The numbers F_n and L_n are considered as special cases of general functions first studied in great detail by Lucas [1]. Let us observe that we may set the Fibonacci and Lucas numbers [2] by

$$F_n = \frac{a^n - b^n}{a - b}, \qquad L_n = a^n + b^n, \tag{1.2}$$

where

$$a = \frac{1}{2}(1 + \sqrt{5}), \qquad b = \frac{1}{2}(1 - \sqrt{5}). \tag{1.3}$$

The very general functions studied by Lucas and generalized by Bell [3], [4], are essentially the F_n and L_n defined by (1.2) with a, b being the roots of the quadratic equation $x^2 = Px - Q$ so that $a + b = P$ and $ab = Q$.

In this paper we get convolution sums formulae almost based on Eqs. (1.1), (1.2), and (1.3), which implies that every proof is simple:

Theorem 1.1. *Let $n, k \in \mathbb{N}$. Then*

(a)

$$\sum_{m=0}^{n} \binom{n}{m} L_{km} L_{n-km} = 2^n L_n + (-1)^{(k-1)n} L_k^n L_{(k-1)n},$$

(b)

$$\sum_{m=0}^{n} \binom{n}{m} L_{km} F_{n-km} = 2^n F_n - (-1)^{(k-1)n} L_k^n F_{(k-1)n},$$

(c)

$$\sum_{m=0}^{n} \binom{n}{m} F_{km} F_{n-km} = \frac{2^n}{5} L_n - \frac{(-1)^{(k-1)n}}{5} L_k^n L_{(k-1)n},$$

(d)

$$\sum_{m=0}^{n} \binom{n}{m} F_{km} L_{n-km} = 2^n F_n + (-1)^{(k-1)n} L_k^n F_{(k-1)n}.$$

Theorem 1.2. *Let $n, k \in \mathbb{N}$. Then*

(a)

$$\sum_{m=0}^{n} L_{km} L_{n-km}$$
$$= (n+1) L_n + \frac{(-1)^{(k-1)n}}{\sum_{i=0}^{k-1} a^{k-1-i} b^i} F_{(2k-1)n+k} + \frac{1}{\sum_{i=0}^{k-1} a^{k-1-i} b^i} F_{n+k},$$

(b)

$$\sum_{m=0}^{n} L_{km} F_{n-km}$$
$$= (n+1) F_n - \frac{(-1)^{(k-1)n}}{5 \sum_{i=0}^{k-1} a^{k-1-i} b^i} L_{(2k-1)n+k} + \frac{1}{5 \sum_{i=0}^{k-1} a^{k-1-i} b^i} L_{n+k},$$

(c)

$$\sum_{m=0}^{n} F_{km}F_{n-km} = \frac{1}{5}(n+1)L_n - \frac{(-1)^{(k-1)n}}{5\sum_{i=0}^{k-1} a^{k-1-i}b^i} F_{(2k-1)n+k} - \frac{1}{5\sum_{i=0}^{k-1} a^{k-1-i}b^i} F_{n+k},$$

(d)

$$\sum_{m=0}^{n} F_{km}L_{n-km} = (n+1)F_n + \frac{(-1)^{(k-1)n}}{5\sum_{i=0}^{k-1} a^{k-1-i}b^i} L_{(2k-1)n+k} - \frac{1}{5\sum_{i=0}^{k-1} a^{k-1-i}b^i} L_{n+k}.$$

Theorem 1.3. *Let $n \in \mathbb{N}$. Then*

(a)

$$\sum_{k=0}^{n}\sum_{m=0}^{k} L_m L_{k-m} L_{n-k} = \frac{(n+1)(n+2)}{2} L_n + 3(n+2)F_{n+1},$$

(b)

$$\sum_{k=0}^{n}\sum_{m=0}^{k} F_m F_{k-m} L_{n-k} = \frac{(n+1)(n+2)}{10} L_n - \frac{n+2}{5} F_{n+1}.$$

2 Proofs of Theorem 1.1, Theorem 1.2, and Theorem 1.3

Proof of Theorem 1.1. Since the proofs are similar and so we only prove part (a). From the definition of L_n in (1.2) we deduce that

$$\begin{aligned}
&\sum_{m=0}^{n} \binom{n}{m} L_{km}L_{n-km} \\
&= \sum_{m=0}^{n} \binom{n}{m} \left(a^{km} + b^{km}\right)\left(a^{n-km} + b^{n-km}\right) \\
&= \sum_{m=0}^{n} \binom{n}{m} \left(a^n + a^{km}b^{n-km} + b^{km}a^{n-km} + b^n\right) \\
&= a^n \sum_{m=0}^{n} \binom{n}{m} + b^n \sum_{m=0}^{n} \binom{n}{m} (\frac{a}{b})^{km} + a^n \sum_{m=0}^{n} \binom{n}{m} (\frac{b}{a})^{km} \\
&\quad + b^n \sum_{m=0}^{n} \binom{n}{m}
\end{aligned}$$

$$
\begin{aligned}
&= a^n \cdot 2^n + b^n(1 + (\frac{a}{b})^k)^n + a^n(1 + (\frac{b}{a})^k)^n + b^n \cdot 2^n \\
&= 2^n a^n + b^n \cdot \frac{1}{b^{kn}}(b^k + a^k)^n + a^n \cdot \frac{1}{a^{kn}}(a^k + b^k)^n + 2^n b^n \\
&= 2^n(a^n + b^n) + (a^k + b^k)^n(\frac{1}{b^{(k-1)n}} + \frac{1}{a^{(k-1)n}}) \\
&= 2^n L_n + (a^k + b^k)^n \cdot \frac{a^{(k-1)n} + b^{(k-1)n}}{(ab)^{(k-1)n}} \\
&= 2^n L_n + (a^k + b^k)^n \cdot (-1)^{(k-1)n}\left(a^{(k-1)n} + b^{(k-1)n}\right) \\
&= 2^n L_n + L_k^n(-1)^{(k-1)n} L_{(k-1)n}
\end{aligned}
$$

since Eq. (1.3) shows that

$$ab = \frac{1}{2}(1 + \sqrt{5}) \cdot \frac{1}{2}(1 - \sqrt{5}) = -1. \tag{2.1}$$

□

Corollary 2.1. *Let $n \in \mathbb{N}$. Then*

(a)

$$\sum_{m=0}^{n} \binom{n}{m} L_m L_{n-m} = 2^n L_n + 2,$$

(b)

$$\sum_{m=0}^{n} \binom{n}{m} L_{2m} L_{n-2m} = (2^n + (-3)^n)\, L_n,$$

(c)

$$\sum_{m=0}^{n} \binom{n}{m} L_{3m} L_{n-3m} = 2^n L_n + 4^n L_{2n},$$

(d)

$$\sum_{m=0}^{n} \binom{n}{m} L_{4m} L_{n-4m} = 2^n L_n + (-7)^n L_{3n}.$$

Proof. By the recurrence definition of Lucas numbers we have $L_0 = 2$, $L_1 = 1$, $L_2 = 3$, $L_3 = 4$, and $L_4 = 7$. Putting $k = 1, 2, 3, 4$ in Theorem 1.1 (a) we have

(a)

$$\sum_{m=0}^{n} \binom{n}{m} L_m L_{n-m} = 2^n L_n + L_1^n L_0 = 2^n L_n + 2.$$

(b)

$$\sum_{m=0}^{n} \binom{n}{m} L_{2m} L_{n-2m} = 2^n L_n + (-1)^n L_2^n L_n = 2^n L_n + (-3)^n L_n.$$

(c)

$$\sum_{m=0}^{n}\binom{n}{m}L_{3m}L_{n-3m} = 2^nL_n + (-1)^{2n}L_3^nL_{2n} = 2^nL_n + 4^nL_{2n},$$

(d)

$$\sum_{m=0}^{n}\binom{n}{m}L_{4m}L_{n-4m} = 2^nL_n + (-1)^{3n}L_4^nL_{3n} = 2^nL_n + (-7)^nL_{3n}.$$

□

Example 2.2. *Let us put $k = 1, 2, 3,$ and 4 in Theorem 1.1 (b), (c), and (d). Then we have the following tables :*

Table 1. $k = 1, 2, 3,$ and 4 in Theorem 1.1 (b)

k	*Convolution sum*	*Formula*
1	$\sum_{m=0}^{n}\binom{n}{m}L_mF_{n-m}$	2^nF_n
2	$\sum_{m=0}^{n}\binom{n}{m}L_{2m}F_{n-2m}$	$(2^n - (-3)^n)F_n$
3	$\sum_{m=0}^{n}\binom{n}{m}L_{3m}F_{n-3m}$	$2^nF_n - 4^nF_{2n}$
4	$\sum_{m=0}^{n}\binom{n}{m}L_{4m}F_{n-4m}$	$2^nF_n - (-7)^nF_{3n}$

Table 2. $k = 1, 2, 3,$ and 4 in Theorem 1.1 (c)

k	*Convolution sum*	*Formula*
1	$\sum_{m=0}^{n}\binom{n}{m}F_mF_{n-m}$	$\frac{2^n}{5}L_n - \frac{2}{5}$
2	$\sum_{m=0}^{n}\binom{n}{m}F_{2m}F_{n-2m}$	$\frac{1}{5}(2^n - (-3)^n)L_n$
3	$\sum_{m=0}^{n}\binom{n}{m}F_{3m}F_{n-3m}$	$\frac{2^n}{5}L_n - \frac{4^n}{5}L_{2n}$
4	$\sum_{m=0}^{n}\binom{n}{m}F_{4m}F_{n-4m}$	$\frac{2^n}{5}L_n - \frac{(-7)^n}{5}L_{3n}$

Table 3. $k = 2, 3,$ and 4 in Theorem 1.1 (d)

k	*Convolution sum*	*Formula*
2	$\sum_{m=0}^{n}\binom{n}{m}F_{2m}L_{n-2m}$	$(2^n + (-3)^n)\,F_n$
3	$\sum_{m=0}^{n}\binom{n}{m}F_{3m}L_{n-3m}$	$2^nF_n + 4^nF_{2n}$
4	$\sum_{m=0}^{n}\binom{n}{m}F_{4m}L_{n-4m}$	$2^nF_n + (-7)^nF_{3n}$

Proof of Theorem 1.2. Since the proof is similar thus we prove only (a). Now by (1.2), (2.1) and the geometric series we have

$$\begin{aligned}
&\sum_{m=0}^{n} L_{km}L_{n-km} \\
&= \sum_{m=0}^{n}(a^{km}+b^{km})(a^{n-km}+b^{n-km}) \\
&= \sum_{m=0}^{n}(a^{n}+a^{km}b^{n-km}+b^{km}a^{n-km}+b^{n}) \\
&= (a^n+b^n)\sum_{m=0}^{n}1+b^n\sum_{m=0}^{n}(\frac{a}{b})^{km}+a^n\sum_{m=0}^{n}(\frac{b}{a})^{km} \\
&= L_n(n+1)+b^n\cdot\frac{(\frac{a}{b})^{k(n+1)}-1}{(\frac{a}{b})^{k}-1}+a^n\cdot\frac{1-(\frac{b}{a})^{k(n+1)}}{1-(\frac{b}{a})^{k}} \\
&= (n+1)L_n+b^n\cdot\frac{b^k(a^{k(n+1)}-b^{k(n+1)})}{b^{k(n+1)}(a^k-b^k)}+a^n\cdot\frac{a^k(a^{k(n+1)}-b^{k(n+1)})}{a^{k(n+1)}(a^k-b^k)} \\
&= (n+1)L_n+\frac{1}{b^{(k-1)n}}\cdot\frac{a^{k(n+1)}-b^{k(n+1)}}{a^k-b^k} \\
&\qquad +\frac{1}{a^{(k-1)n}}\cdot\frac{a^{k(n+1)}-b^{k(n+1)}}{a^k-b^k} \\
&= (n+1)L_n+\frac{a^{k(n+1)}-b^{k(n+1)}}{a^k-b^k}\left(\frac{1}{b^{(k-1)n}}+\frac{1}{a^{(k-1)n}}\right) \\
&= (n+1)L_n+\frac{a^{k(n+1)}-b^{k(n+1)}}{a^k-b^k}\cdot\frac{a^{(k-1)n}+b^{(k-1)n}}{(ab)^{(k-1)n}} \\
&= (n+1)L_n+\frac{(-1)^{(k-1)n}}{a^k-b^k}\cdot\left(a^{k(n+1)}-b^{k(n+1)}\right)\left(a^{(k-1)n}+b^{(k-1)n}\right).
\end{aligned} \tag{2.2}$$

Then the second term in the above identity can be written as

$$\begin{aligned}
&\frac{(-1)^{(k-1)n}}{a^k-b^k}\cdot\left(a^{k(n+1)}-b^{k(n+1)}\right)\left(a^{(k-1)n}+b^{(k-1)n}\right) \\
&= \frac{(-1)^{(k-1)n}}{a^k-b^k}\cdot\Big(a^{2kn+k-n}+a^{k(n+1)}b^{(k-1)n}-b^{k(n+1)}a^{(k-1)n} \\
&\qquad -b^{2kn+k-n}\Big) \\
&= \frac{(-1)^{(k-1)n}}{a^k-b^k}\cdot\left\{(a^{2kn+k-n}-b^{2kn+k-n})-(ab)^{k+kn}(a^{-n-k}-b^{-n-k})\right\} \\
&= \frac{(-1)^{(k-1)n}}{\sum_{i=0}^{k-1}a^{k-1-i}b^i}\Bigg(\frac{a^{2kn+k-n}-b^{2kn+k-n}}{a-b} \\
&\qquad -(-1)^{k(n+1)}\cdot\frac{a^{-n-k}-b^{-n-k}}{a-b}\Bigg)
\end{aligned}$$

and so by (1.1) we deduce that

$$
\begin{aligned}
&\frac{(-1)^{(k-1)n}}{a^k-b^k}\cdot\left(a^{k(n+1)}-b^{k(n+1)}\right)\left(a^{(k-1)n}+b^{(k-1)n}\right)\\
&=\frac{(-1)^{(k-1)n}}{\sum_{i=0}^{k-1}a^{k-1-i}b^i}\left(F_{2kn+k-n}-(-1)^{k(n+1)}F_{-(n+k)}\right)\\
&=\frac{(-1)^{(k-1)n}}{\sum_{i=0}^{k-1}a^{k-1-i}b^i}\left(F_{2kn+k-n}-(-1)^{k(n+1)}\cdot(-1)^{n+k+1}F_{n+k}\right)\\
&=\frac{(-1)^{(k-1)n}}{\sum_{i=0}^{k-1}a^{k-1-i}b^i}F_{2kn+k-n}+\frac{1}{\sum_{i=0}^{k-1}a^{k-1-i}b^i}F_{n+k}.
\end{aligned}
\tag{2.3}
$$

Applying (2.3) to (2.2) we obtain

$$
\begin{aligned}
&\sum_{m=0}^{n}L_{km}L_{n-km}\\
&=(n+1)L_n+\frac{(-1)^{(k-1)n}}{\sum_{i=0}^{k-1}a^{k-1-i}b^i}F_{(2k-1)n+k}+\frac{1}{\sum_{i=0}^{k-1}a^{k-1-i}b^i}F_{n+k}.
\end{aligned}
$$

□

Example 2.3. *Let us consider $k = 1, 2, 3,$ and 4 in Theorem 1.2 (a), (b), (c), and (d). Then we have the following tables:*

Table 4. $k = 1, 2, 3,$ and 4 in Theorem 1.2 (a)

k	*Convolution sum*	*Formula*
1	$\sum_{m=0}^{n}L_mL_{n-m}$	$(n+1)L_n+2F_{n+1}$
2	$\sum_{m=0}^{n}L_{2m}L_{n-2m}$	$(n+1)L_n+(-1)^nF_{3n+2}+F_{n+2}$
3	$\sum_{m=0}^{n}L_{3m}L_{n-3m}$	$(n+1)L_n+\frac{1}{2}F_{5n+3}+\frac{1}{2}F_{n+3}$
4	$\sum_{m=0}^{n}L_{4m}L_{n-4m}$	$(n+1)L_n+\frac{(-1)^n}{3}F_{7n+4}+\frac{1}{3}F_{n+4}$

Table 5. $k = 1, 2, 3,$ and 4 in Theorem 1.2 (b)

k	*Convolution sum*	*Formula*
1	$\sum_{m=0}^{n}L_mF_{n-m}$	$(n+1)F_n$
2	$\sum_{m=0}^{n}L_{2m}F_{n-2m}$	$(n+1)F_n-\frac{(-1)^n}{5}L_{3n+2}+\frac{1}{5}L_{n+2}$
3	$\sum_{m=0}^{n}L_{3m}F_{n-3m}$	$(n+1)F_n-\frac{1}{10}L_{5n+3}+\frac{1}{10}L_{n+3}$
4	$\sum_{m=0}^{n}L_{4m}F_{n-4m}$	$(n+1)F_n-\frac{(-1)^n}{15}L_{7n+4}+\frac{1}{15}L_{n+4}$

Table 6. $k = 1, 2, 3,$ **and** 4 **in Theorem 1.2 (c)**

k	*Convolution sum*	*Formula*
1	$\sum_{m=0}^{n} F_m F_{n-m}$	$\frac{1}{5}(n+1)L_n - \frac{2}{5}F_{n+1}$
2	$\sum_{m=0}^{n} F_{2m} F_{n-2m}$	$\frac{1}{5}(n+1)L_n - \frac{(-1)^n}{5}F_{3n+2} - \frac{1}{5}F_{n+2}$
3	$\sum_{m=0}^{n} F_{3m} F_{n-3m}$	$\frac{1}{5}(n+1)L_n - \frac{1}{10}F_{5n+3} - \frac{1}{10}F_{n+3}$
4	$\sum_{m=0}^{n} F_{4m} F_{n-4m}$	$\frac{1}{5}(n+1)L_n - \frac{(-1)^n}{15}F_{7n+4} - \frac{1}{15}F_{n+4}$

Table 7. $k = 2, 3,$ **and** 4 **in Theorem 1.2 (d)**

k	*Convolution sum*	*Formula*
2	$\sum_{m=0}^{n} F_{2m} L_{n-2m}$	$(n+1)F_n + \frac{(-1)^n}{5}L_{3n+2} - \frac{1}{5}L_{n+2}$
3	$\sum_{m=0}^{n} F_{3m} L_{n-3m}$	$(n+1)F_n + \frac{1}{10}L_{5n+3} - \frac{1}{10}L_{n+3}$
4	$\sum_{m=0}^{n} F_{4m} L_{n-4m}$	$(n+1)F_n + \frac{(-1)^n}{15}L_{7n+4} - \frac{1}{15}L_{n+4}$

Proof of Theorem 1.3. (a) Table 4 enables us to deduce that

$$
\begin{aligned}
&\sum_{k=0}^{n}\sum_{m=0}^{k} L_m L_{k-m} L_{n-k} \\
&= \sum_{k=0}^{n}\left(\sum_{m=0}^{k} L_m L_{k-m}\right) L_{n-k} \\
&= \sum_{k=0}^{n}\left((k+1)L_k + 2F_{k+1}\right) L_{n-k} \\
&= \sum_{k=0}^{n}(k+1)L_k L_{n-k} + 2\sum_{k=0}^{n} F_{k+1}L_{n-k} \\
&= \sum_{k=0}^{n} kL_k L_{n-k} + \sum_{k=0}^{n} L_k L_{n-k} + 2\sum_{k=0}^{n} F_{k+1}L_{n-k}.
\end{aligned}
\tag{2.4}
$$

Then we note that

$$\sum_{k=0}^{n} kL_k L_{n-k} = \sum_{k=0}^{n}(n-k)L_{n-k}L_k$$

and so

$$\sum_{k=0}^{n} kL_k L_{n-k} = \frac{n}{2}\sum_{k=0}^{n} L_k L_{n-k}. \tag{2.5}$$

Applying (2.5) to (2.4) and using Table 4 and 5, we have

$$\begin{aligned}
&\sum_{k=0}^{n}\sum_{m=0}^{k} L_m L_{k-m} L_{n-k} \\
&= \frac{n}{2}\sum_{k=0}^{n} L_k L_{n-k} + \sum_{k=0}^{n} L_k L_{n-k} + 2\sum_{k=0}^{n} F_{n+1-k} L_k \\
&= \frac{n}{2}\left((n+1)L_n + 2F_{n+1}\right) + \left((n+1)L_n + 2F_{n+1}\right) + 2(n+2)F_{n+1} \\
&= \frac{(n+1)(n+2)}{2} L_n + 3(n+2)F_{n+1}.
\end{aligned}$$

(b) By Table 6 we expand as follows :

$$\begin{aligned}
&\sum_{k=0}^{n}\sum_{m=0}^{k} F_m F_{k-m} L_{n-k} \\
&= \sum_{k=0}^{n}\left(\sum_{m=0}^{k} F_m F_{k-m}\right) L_{n-k} \\
&= \sum_{k=0}^{n} \frac{1}{5}\left((k+1)L_k - 2F_{k+1}\right) L_{n-k} \qquad (2.6)\\
&= \frac{1}{5}\sum_{k=0}^{n}(k+1)L_k L_{n-k} - \frac{2}{5}\sum_{k=0}^{n} F_{k+1} L_{n-k} \\
&= \frac{1}{5}\sum_{k=0}^{n} kL_k L_{n-k} + \frac{1}{5}\sum_{k=0}^{n} L_k L_{n-k} - \frac{2}{5}\sum_{k=0}^{n} F_{k+1} L_{n-k}.
\end{aligned}$$

After applying (2.5) to (2.6) and employing Table 4 and 5, we can write

$$\begin{aligned}
&\sum_{k=0}^{n}\sum_{m=0}^{k} F_m F_{k-m} L_{n-k} \\
&= \frac{1}{5}\cdot\frac{n}{2}\sum_{k=0}^{n} L_k L_{n-k} + \frac{1}{5}\sum_{k=0}^{n} L_k L_{n-k} - \frac{2}{5}\sum_{k=0}^{n} F_{n+1-k} L_k \\
&= \frac{n}{10}\left((n+1)L_n + 2F_{n+1}\right) + \frac{1}{5}\left((n+1)L_n + 2F_{n+1}\right) - \frac{2}{5}(n+2)F_{n+1} \\
&= \frac{(n+1)(n+2)}{10} L_n - \frac{n+2}{5} F_{n+1}.
\end{aligned}$$

□

3 Conclusion

For the Fibonacci number and Lucas number given by

$$F_n = \frac{a^n - b^n}{a-b} \qquad \text{and} \qquad L_n = a^n + b^n,$$

we construct some convolutions sums and get their generalized formulae.

Competing Interests

Author has declared that no competing interests exist.

References

[1] Lucas E. Théorie des fonctions numériques simplement périodiques. Amer. J. Math. 1878;1(2):184-196.

[2] Gould HW. Generating functions for products of powers of fibonacci numbers. Fibonacci Quarterly. 1963;2(1):1-16.

[3] Bell ET. A revision of the algebra of Lucas functions. Annals of Math. 1935;2(36):733-742.

[4] Bell ET. Arithmetical theorems on Lucas functions and Tchebycheff polynomials. Amer. J. Math. 1935;(57):781-788.
