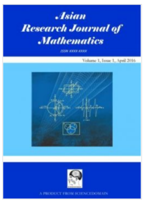

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The Product-to-sum Expressing with a Divisor Function $\sigma_3(n)$

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

A formula expressing an infinite product as an infinite sum is called a product-to-sum identity. In this paper we try to consider a special product-to-sum as

$$q^r \prod_{n=1}^{\infty} (1-q^n)^a (1-q^{2n})^b (1-q^{4n})^c = u + \sum_{n=1}^{\infty} \left\{ x\sigma_3(n) + y\sigma_3(\frac{n}{2}) + z\sigma_3(\frac{n}{4}) \right\} q^n$$

and so for integers r, u, a, b, c, x, y, and z we deduce all solutions of (r, u, a, b, c, x, y, z) with $r \geq 0$.

Keywords: Divisor functions; infinite product sums; product-to-sum.

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1 Introduction

Let q be a complex variable with $|q| < 1$. As in ([1], p. 850), ([2], p. 6), the theta function is defined by

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}. \tag{1.1}$$

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Based on Berndt ([2], p. 119–120), it is essential that

$$x = x(q) := 1 - \frac{\varphi^4(-q)}{\varphi^4(q)}, \quad z = z(q) := \varphi^2(q). \tag{1.2}$$

And Ramanujan gave in his notebook the following formulae proved in ([3], p. 126–129), ([4], p. 44):

$$M(q) = (1 + 14x + x^2)z^4, \tag{1.3}$$

$$M(q^2) = (1 - x + x^2)z^4, \tag{1.4}$$

$$M(q^4) = (1 - x + \frac{1}{16}x^2)z^4. \tag{1.5}$$

Moreover we note that

$$\varphi(q) = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^5}{(1-q^n)^2(1-q^{4n})^2} \qquad \text{and} \qquad xz^2 = 16q\prod_{n=1}^{\infty}\frac{(1-q^{4n})^8}{(1-q^{2n})^4} \tag{1.6}$$

in ([5] (11), p. 339). It is well-known that Jacobi's triple product identity ([6], Vol. I, p. 49–239) as

$$\prod_{n=1}^{\infty}(1-q^{2n})(1+aq^{2n-1})(1+a^{-1}q^{2n-1}) = \sum_{n=-\infty}^{\infty} a^n q^{n^2}$$

for all complex numbers a and q with nonzero a and $|q| < 1$. Using Jacobi's triple product identity, K. S. Williams generalizes the product-to-sum formula states that

Proposition 1.1. *(See [5] Theorem 4) Let r, u, a, b, c, x, y, and z be integers with $r \geq 0$ such that*

$$q^r \prod_{n=1}^{\infty}(1-q^n)^a(1-q^{2n})^b(1-q^{4n})^c = u + \sum_{n=1}^{\infty}\left\{x\sigma_1(n) + y\sigma_1(\frac{n}{2}) + z\sigma_1(\frac{n}{4})\right\}q^n.$$

Then

$$\begin{aligned}(r,u,a,b,c,x,y,z) =& (0,1,0,0,0,0,0,0), \quad (0,1,-8,20,-8,8,0,-32),\\ & (0,1,8,-4,0,-8,48,-64), \quad \textit{or} \quad (1,0,0,-4,8,1,-3,2).\end{aligned}$$

As usual we define that $\sigma_s(n)$ is the sum of s-th power of the divisors of n with nonnegative integer s.

In this article we are also interested in the product-to-sum, for example,

$$\prod_{n=1}^{\infty}(1-q^n)^{-16}(1-q^{2n})^{40}(1-q^{4n})^{-16} = 1 + \sum_{n=1}^{\infty}\left\{16\sigma_3(n) - 32\sigma_3(\frac{n}{2}) + 256\sigma_3(\frac{n}{4})\right\}q^n,$$

$$q\prod_{n=1}^{\infty}(1-q^n)^{8}(1-q^{2n})^{-8}(1-q^{4n})^{8} = \sum_{n=1}^{\infty}\left\{\sigma_3(n) - 17\sigma_3(\frac{n}{2}) + 16\sigma_3(\frac{n}{4})\right\}q^n,$$

(see Theorem 2.1 and Theorem 2.2), and etc. Inspiration from Proposition 1.1, that is, replacing $\sigma_1(n)$ by $\sigma_3(n)$ leads us to obtain the following theorem :

Theorem 1.1. *Let r, u, a, b, c, x, y, and z be integers with $r \geq 0$ such that*

$$q^r \prod_{n=1}^{\infty}(1-q^n)^a(1-q^{2n})^b(1-q^{4n})^c = u + \sum_{n=1}^{\infty}\left\{x\sigma_3(n) + y\sigma_3(\frac{n}{2}) + z\sigma_3(\frac{n}{4})\right\}q^n. \tag{1.7}$$

Then we have

$$\begin{aligned}(r,u,a,b,c,x,y,z) =&(0,1,0,0,0,0,0,0), \quad (0,1,-16,40,-16,16,-32,256),\\ &(0,1,16,-8,0,-16,256,0), \quad (1,0,-8,16,0,1,-1,0),\\ &(1,0,8,-8,8,1,-17,16), \quad \textit{or} \quad (2,0,0,-8,16,0,1,-1).\end{aligned}$$

2 Proof of Theorem 2.1, Theorem 2.2, Theorem 2.3, and Theorem 2.4

Let q be a complex variable satisfying $|q| < 1$. Then the Eisenstein series $M(q)$ is given by

$$M(q) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n \tag{2.1}$$

in ([7], p. 318), ([8], eqn. (25)), ([9], p. 389).

Remark 2.1. Let us find rational numbers α, β, and γ such that

$$\alpha M(q) + \beta M(q^2) + \gamma M(q^4) = z^4.$$

By (1.3), (1.4), and (1.5) the above identity can be written as

$$\begin{aligned}&\alpha M(q) + \beta M(q^2) + \gamma M(q^4)\\ &= \alpha(1+14x+x^2)z^4 + \beta(1-x+x^2)z^4 + \gamma(1-x+\frac{1}{16}x^2)z^4\\ &= (\alpha+\beta+\gamma)\, z^4 + (14\alpha-\beta-\gamma)\, xz^4 + \left(\alpha+\beta+\frac{1}{16}\gamma\right)x^2z^4\\ &= z^4,\end{aligned}$$

which shows that

$$\alpha+\beta+\gamma = 1, \quad 14\alpha-\beta-\gamma = 0, \quad \alpha+\beta+\frac{1}{16}\gamma = 0$$

and so

$$\alpha = \frac{1}{15}, \quad \beta = -\frac{2}{15}, \quad \gamma = \frac{16}{15}.$$

Thus we conclude that

$$\frac{1}{15}M(q) - \frac{2}{15}M(q^2) + \frac{16}{15}M(q^4) = z^4. \tag{2.2}$$

From (1.2), (1.6), (2.1), and (2.2) we have

$$\begin{aligned}
&\frac{1}{15}M(q) - \frac{2}{15}M(q^2) + \frac{16}{15}M(q^4) \\
&= z^4 \\
&= \varphi^8(q) \\
&= \left\{ \prod_{n=1}^{\infty} \frac{(1-q^{2n})^5}{(1-q^n)^2(1-q^{4n})^2} \right\}^8 \\
&= \prod_{n=1}^{\infty} (1-q^{2n})^{40}(1-q^n)^{-16}(1-q^{4n})^{-16} \\
&= \frac{1}{15}\left(1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n\right) - \frac{2}{15}\left(1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^{2n}\right) \\
&\quad + \frac{16}{15}\left(1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^{4n}\right) \\
&= 1 + 16\sum_{n=1}^{\infty}\sigma_3(n)q^n - 32\sum_{n=1}^{\infty}\sigma_3(n)q^{2n} + 256\sum_{n=1}^{\infty}\sigma_3(n)q^{4n}
\end{aligned}$$

which summarizes that

$$\begin{aligned}
&\prod_{n=1}^{\infty}(1-q^n)^{-16}(1-q^{2n})^{40}(1-q^{4n})^{-16} \\
&= 1 + \sum_{n=1}^{\infty}\left\{16\sigma_3(n) - 32\sigma_3(\frac{n}{2}) + 256\sigma_3(\frac{n}{4})\right\}q^n.
\end{aligned} \tag{2.3}$$

Finally we are willing to find the similar form as (2.3) and so we deduce Theorem 2.1.

Theorem 2.1. *If a, b, c, x, y, z are integers with $(a,b,c) \neq (0,0,0)$ such that*

$$\prod_{n=1}^{\infty}(1-q^n)^a(1-q^{2n})^b(1-q^{4n})^c = 1 + \sum_{n=1}^{\infty}\left\{x\sigma_3(n) + y\sigma_3(\frac{n}{2}) + z\sigma_3(\frac{n}{4})\right\}q^n \tag{2.4}$$

holds, then

$$(a,b,c,x,y,z) = (-16, 40, -16, 16, -32, 256) \quad or \quad (16, -8, 0, -16, 256, 0).$$

Proof. Using MAPLE, we find that the left hand side of (2.4) is

$$\prod_{n=1}^{\infty}(1-q^n)^a(1-q^{2n})^b(1-q^{4n})^c$$
$$= 1 - aq + \left(\frac{1}{2}a^2 - \frac{3}{2}a - b\right)q^2 + \left(-\frac{1}{6}a^3 + \frac{3}{2}a^2 - \frac{4}{3}a + ab\right)q^3$$
$$+ \left(\frac{1}{24}a^4 - \frac{3}{4}a^3 + \frac{59}{24}a^2 - \frac{7}{4}a - \frac{1}{2}a^2b + \frac{3}{2}ab + \frac{1}{2}b^2 - \frac{3}{2}b - c\right)q^4$$
$$+ \left(-\frac{1}{120}a^5 + \frac{1}{4}a^4 - \frac{43}{24}a^3 + \frac{15}{4}a^2 - \frac{6}{5}a + \frac{1}{6}a^3b - \frac{3}{2}a^2b + \frac{17}{6}ab - \frac{1}{2}ab^2 + ac\right)q^5$$
$$+ \left(\frac{1}{720}a^6 - \frac{1}{16}a^5 + \frac{113}{144}a^4 - \frac{55}{16}a^3 + \frac{1697}{360}a^2 - 2a - \frac{1}{24}a^4b + \frac{3}{4}a^3b - \frac{77}{24}a^2b\right.$$
$$\left.+4ab + \frac{1}{4}a^2b^2 - \frac{3}{4}ab^2 - \frac{1}{6}b^3 + \frac{3}{2}b^2 - \frac{4}{3}b - \frac{1}{2}a^2c + \frac{3}{2}ac + bc\right)q^6$$
$$+ \left(-\frac{1}{5040}a^7 + \frac{1}{80}a^6 - \frac{35}{144}a^5 + \frac{89}{48}a^4 - \frac{2021}{360}a^3 + \frac{92}{15}a^2 - \frac{8}{7}a + \frac{1}{120}a^5b - \frac{1}{4}a^4b\right.$$
$$+ \frac{49}{24}a^3b - 6a^2b + \frac{68}{15}ab - \frac{1}{12}a^3b^2 + \frac{3}{4}a^2b^2 - \frac{13}{6}ab^2 + \frac{1}{6}ab^3 + \frac{1}{6}a^3c - \frac{3}{2}a^2c$$
$$\left.+\frac{4}{3}ac - abc\right)q^7 + \cdots .$$

And the right hand side of (2.4) is

$$1 + \sum_{n=1}^{\infty}\left\{x\sigma_3(n) + y\sigma_3(\frac{n}{2}) + z\sigma_3(\frac{n}{4})\right\}q^n$$
$$= 1 + xq + (9x+y)\,q^2 + 28xq^3 + (73x+9y+z)\,q^4 + 126xq^5 + (252x+28y)\,q^6$$
$$+ 344xq^7 + \cdots .$$

Equating coefficients of (2.4), we have

$$-a = x, \tag{2.5}$$

$$\frac{1}{2}a^2 - \frac{3}{2}a - b = 9x + y, \tag{2.6}$$

$$-\frac{1}{6}a^3 + \frac{3}{2}a^2 - \frac{4}{3}a + ab = 28x, \tag{2.7}$$

$$\frac{1}{24}a^4 - \frac{3}{4}a^3 + \frac{59}{24}a^2 - \frac{7}{4}a - \frac{1}{2}a^2b + \frac{3}{2}ab + \frac{1}{2}b^2 - \frac{3}{2}b - c = 73x + 9y + z, \tag{2.8}$$

$$-\frac{1}{120}a^5 + \frac{1}{4}a^4 - \frac{43}{24}a^3 + \frac{15}{4}a^2 - \frac{6}{5}a + \frac{1}{6}a^3b - \frac{3}{2}a^2b + \frac{17}{6}ab - \frac{1}{2}ab^2 + ac = 126x, \tag{2.9}$$

$$\begin{aligned}&\frac{1}{720}a^6 - \frac{1}{16}a^5 + \frac{113}{144}a^4 - \frac{55}{16}a^3 + \frac{1697}{360}a^2 - 2a - \frac{1}{24}a^4b + \frac{3}{4}a^3b - \frac{77}{24}a^2b\\ &+ 4ab + \frac{1}{4}a^2b^2 - \frac{3}{4}ab^2 - \frac{1}{6}b^3 + \frac{3}{2}b^2 - \frac{4}{3}b - \frac{1}{2}a^2c + \frac{3}{2}ac + bc = 252x + 28y,\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}
&-\frac{1}{5040}a^7+\frac{1}{80}a^6-\frac{35}{144}a^5+\frac{89}{48}a^4-\frac{2021}{360}a^3+\frac{92}{15}a^2-\frac{8}{7}a+\frac{1}{120}a^5b-\frac{1}{4}a^4b\\
&+\frac{49}{24}a^3b-6a^2b+\frac{68}{15}ab-\frac{1}{12}a^3b^2+\frac{3}{4}a^2b^2-\frac{13}{6}ab^2+\frac{1}{6}ab^3+\frac{1}{6}a^3c-\frac{3}{2}a^2c\\
&+\frac{4}{3}ac-abc=344x.
\end{aligned} \tag{2.11}$$

Now suppose that $a \neq 0$. The case that $a = 0$ will be treated at the end of the proof. From (2.5) and (2.7) we obtain

$$b=\frac{1}{6}a^2-\frac{3}{2}a-\frac{80}{3}. \tag{2.12}$$

By (2.5), (2.9), and (2.12), we deduce that

$$c=-\frac{1}{180}a^4+\frac{7}{36}a^2+\frac{1}{2}a+\frac{13784}{45}. \tag{2.13}$$

Applying (2.5), (2.12), and (2.13) to (2.11) we have

$$-\frac{1}{2835}a^6+\frac{4}{135}a^4-\frac{64}{135}a^2+\frac{11616256}{2835}=0$$

so that

$$a^6-84a^4+1344a^2-11616256=0.$$

The above equation implies that

$$a(a+16)(a-16)(a^4+172a^2+45376)=0.$$

Since $a^4 + 172a^2 + 45376 > 0$ for an integer a and here assuming that $a \neq 0$, thus the appropriate a are -16 and 16.
If $a = -16$ then by (2.5), (2.12), and (2.13), respectively we obtain

$$x=16, \quad b=40, \quad \text{and} \quad c=-16. \tag{2.14}$$

From (2.6) and (2.14) we have

$$y=-32, \tag{2.15}$$

also by (2.8), (2.14), and (2.15) we note that $z = 256$. So we conclude that

$$(a,b,c,x,y,z)=(-16,40,-16,16,-32,256)$$

which is also found in Eq. (2.3). In a similar manner, when $a = 16$ we obtain

$$(a,b,c,x,y,z)=(16,-8,0,-16,256,0).$$

To confirm the proof we compare some coefficients in

$$X:=\prod_{n=1}^{\infty}(1-q^n)^a(1-q^{2n})^b(1-q^{4n})^c$$

and

$$Y := 1 + \sum_{n=1}^{\infty} \left\{ x\sigma_3(n) + y\sigma_3(\frac{n}{2}) + z\sigma_3(\frac{n}{4}) \right\} q^n.$$

We show two following tables :

Table 1. Coefficients of q^n in X and Y for n ($8 \le n \le 15$) when $(a, b, c, x, y, z) = (-16, 40, -16, 16, -32, 256)$

n	8	9	10	11	12	13	14	15
X	9328	12112	14112	21312	31808	35168	38528	56448
Y	9328	12112	14112	21312	31808	35168	38528	56448

Table 2. Coefficients of q^n in X and Y for n ($8 \le n \le 15$) when $(a, b, c, x, y, z) = (16, -8, 0, -16, 256, 0)$

n	8	9	10	11	12	13	14	15
X	9328	−12112	14112	−21312	31808	−35168	38528	−56448
Y	9328	−12112	14112	−21312	31808	−35168	38528	−56448

Next we turn to the case $a = 0$. Then Eq. (2.4) becomes

$$\prod_{n=1}^{\infty} (1 - q^{2n})^b (1 - q^{4n})^c = 1 + \sum_{n=1}^{\infty} \left\{ x\sigma_3(n) + y\sigma_3(\frac{n}{2}) + z\sigma_3(\frac{n}{4}) \right\} q^n.$$

Equating the coefficients of q on both sides, we obtain that $x = 0$ and so

$$\begin{aligned} \prod_{n=1}^{\infty} (1 - q^{2n})^b (1 - q^{4n})^c &= 1 + \sum_{n=1}^{\infty} \left\{ y\sigma_3(\frac{n}{2}) + z\sigma_3(\frac{n}{4}) \right\} q^n \\ &= 1 + \sum_{n=1}^{\infty} \left\{ y\sigma_3(n) + z\sigma_3(\frac{n}{2}) \right\} q^{2n}. \end{aligned}$$

Substituting q^2 with q, we have

$$\prod_{n=1}^{\infty} (1 - q^{n})^b (1 - q^{2n})^c = 1 + \sum_{n=1}^{\infty} \left\{ y\sigma_3(n) + z\sigma_3(\frac{n}{2}) \right\} q^{n},$$

which implies the above mentioned solution that

$$(a, b, c, x, y, z) = (16, -8, 0, -16, 256, 0).$$

Therefore this completes the proof.

□

Remark 2.2. In a similar manner to Remark 2.1 we can find rational numbers α, β, and γ satisfying that

$$\alpha M(q) + \beta M(q^2) + \gamma M(q^4) = xz^4.$$

By (1.3), (1.4), and (1.5) the above equation is changed as

$$\begin{aligned}&\alpha M(q)+\beta M(q^2)+\gamma M(q^4)\\&=(\alpha+\beta+\gamma)\,z^4+(14\alpha-\beta-\gamma)\,xz^4+\left(\alpha+\beta+\frac{1}{16}\gamma\right)x^2z^4\\&=xz^4,\end{aligned}$$

which leads that

$$\alpha+\beta+\gamma=0,\quad 14\alpha-\beta-\gamma=1,\quad \alpha+\beta+\frac{1}{16}\gamma=0$$

and so

$$\alpha=\frac{1}{15},\quad \beta=-\frac{1}{15},\quad \gamma=0.$$

Thus we conclude that

$$\frac{1}{15}M(q)-\frac{1}{15}M(q^2)=xz^4. \tag{2.16}$$

From (1.2), (1.6), (2.1), and (2.16) we can deduce that

$$\begin{aligned}&\frac{1}{15}M(q)-\frac{1}{15}M(q^2)\\&=xz^4\\&=z^2\cdot xz^2\\&=\varphi^4(q)\cdot xz^2\\&=\left\{\prod_{n=1}^{\infty}\frac{(1-q^{2n})^5}{(1-q^n)^2(1-q^{4n})^2}\right\}^4\cdot 16q\prod_{n=1}^{\infty}\frac{(1-q^{4n})^8}{(1-q^{2n})^4}\\&=16q\prod_{n=1}^{\infty}(1-q^n)^{-8}(1-q^{2n})^{16}\\&=\frac{1}{15}\left(1+240\sum_{n=1}^{\infty}\sigma_3(n)q^n\right)-\frac{1}{15}\left(1+240\sum_{n=1}^{\infty}\sigma_3(n)q^{2n}\right)\\&=16\sum_{n=1}^{\infty}\sigma_3(n)q^n-16\sum_{n=1}^{\infty}\sigma_3(n)q^{2n}\end{aligned}$$

which summarizes that

$$q\prod_{n=1}^{\infty}(1-q^n)^{-8}(1-q^{2n})^{16}=\sum_{n=1}^{\infty}\left\{\sigma_3(n)-\sigma_3(\frac{n}{2})\right\}q^n. \tag{2.17}$$

Also we can obtain another similar form as (2.17) and so we consider Theorem 2.2.

Theorem 2.2. *If a, b, c, x, y, z are integers such that*

$$q\prod_{n=1}^{\infty}(1-q^n)^a(1-q^{2n})^b(1-q^{4n})^c=\sum_{n=1}^{\infty}\left\{x\sigma_3(n)+y\sigma_3(\frac{n}{2})+z\sigma_3(\frac{n}{4})\right\}q^n \tag{2.18}$$

holds, then

$$(a,b,c,x,y,z) = (-8,16,0,1,-1,0) \quad or \quad (8,-8,8,1,-17,16).$$

Proof. Using MAPLE, we find that the left hand side of (2.18) is

$$\begin{aligned}
&q\prod_{n=1}^{\infty}(1-q^n)^a(1-q^{2n})^b(1-q^{4n})^c \\
&= q - aq^2 + \left(\frac{1}{2}a^2 - \frac{3}{2}a - b\right)q^3 + \left(-\frac{1}{6}a^3 + \frac{3}{2}a^2 - \frac{4}{3}a + ab\right)q^4 \\
&\quad + \left(\frac{1}{24}a^4 - \frac{3}{4}a^3 + \frac{59}{24}a^2 - \frac{7}{4}a - \frac{1}{2}a^2b + \frac{3}{2}ab + \frac{1}{2}b^2 - \frac{3}{2}b - c\right)q^5 \\
&\quad + \left(-\frac{1}{120}a^5 + \frac{1}{4}a^4 - \frac{43}{24}a^3 + \frac{15}{4}a^2 - \frac{6}{5}a + \frac{1}{6}a^3b - \frac{3}{2}a^2b + \frac{17}{6}ab - \frac{1}{2}ab^2 + ac\right)q^6 \\
&\quad + \left(\frac{1}{720}a^6 - \frac{1}{16}a^5 + \frac{113}{144}a^4 - \frac{55}{16}a^3 + \frac{1697}{360}a^2 - 2a - \frac{1}{24}a^4b + \frac{3}{4}a^3b - \frac{77}{24}a^2b\right. \\
&\qquad \left. +4ab + \frac{1}{4}a^2b^2 - \frac{3}{4}ab^2 - \frac{1}{6}b^3 + \frac{3}{2}b^2 - \frac{4}{3}b - \frac{1}{2}a^2c + \frac{3}{2}ac + bc\right)q^7 + \cdots .
\end{aligned}$$

The right hand side of (2.18) is

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left\{x\sigma_3(n) + y\sigma_3(\frac{n}{2}) + z\sigma_3(\frac{n}{4})\right\}q^n \\
&= xq + (9x+y)\,q^2 + 28xq^3 + (73x+9y+z)\,q^4 + 126xq^5 + (252x+28y)\,q^6 \\
&\quad + 344xq^7 + \cdots .
\end{aligned}$$

Equating coefficients of (2.18), we observe that

$$1 = x, \tag{2.19}$$

$$-a = 9x + y, \tag{2.20}$$

$$\frac{1}{2}a^2 - \frac{3}{2}a - b = 28x, \tag{2.21}$$

$$-\frac{1}{6}a^3 + \frac{3}{2}a^2 - \frac{4}{3}a + ab = 73x + 9y + z, \tag{2.22}$$

$$\frac{1}{24}a^4 - \frac{3}{4}a^3 + \frac{59}{24}a^2 - \frac{7}{4}a - \frac{1}{2}a^2b + \frac{3}{2}ab + \frac{1}{2}b^2 - \frac{3}{2}b - c = 126x, \tag{2.23}$$

$$-\frac{1}{120}a^5 + \frac{1}{4}a^4 - \frac{43}{24}a^3 + \frac{15}{4}a^2 - \frac{6}{5}a + \frac{1}{6}a^3b - \frac{3}{2}a^2b + \frac{17}{6}ab - \frac{1}{2}ab^2 + ac = 252x + 28y, \tag{2.24}$$

and

$$\begin{aligned}
&\frac{1}{720}a^6 - \frac{1}{16}a^5 + \frac{113}{144}a^4 - \frac{55}{16}a^3 + \frac{1697}{360}a^2 - 2a - \frac{1}{24}a^4b + \frac{3}{4}a^3b - \frac{77}{24}a^2b \\
&\quad + 4ab + \frac{1}{4}a^2b^2 - \frac{3}{4}ab^2 - \frac{1}{6}b^3 + \frac{3}{2}b^2 - \frac{4}{3}b - \frac{1}{2}a^2c + \frac{3}{2}ac + bc = 344x.
\end{aligned} \tag{2.25}$$

From (2.19) and (2.21) we have

$$b = \frac{1}{2}a^2 - \frac{3}{2}a - 28. \tag{2.26}$$

By (2.19), (2.23), and (2.26), we obtain

$$c = -\frac{1}{12}a^4 + \frac{7}{12}a^2 + \frac{1}{2}a + 308. \tag{2.27}$$

Applying (2.19), (2.26), and (2.27) to (2.25) we deduce that

$$(a+8)(a-8)(a^4 + 44a^2 + 2880) = 0.$$

Since $a^4 + 44a^2 + 2880 > 0$ for an integer a, therefore the possible value of a are -8 and 8.
If $a = -8$ then by (2.26) and (2.27), we have

$$b = 16 \quad \text{and} \quad c = 0. \tag{2.28}$$

From (2.19) and (2.20) we obtain

$$y = -1, \tag{2.29}$$

also by (2.19), (2.22), (2.28), and (2.29) we have that $z = 0$. So we conclude that

$$(a, b, c, x, y, z) = (-8, 16, 0, 1, -1, 0)$$

which is also found in Eq. (2.17). Similarly, when $a = 8$ we obtain

$$(a, b, c, x, y, z) = (8, -8, 8, 1, -17, 16).$$

To ensure the proof we compare some coefficients in

$$U := q\prod_{n=1}^{\infty}(1-q^n)^a(1-q^{2n})^b(1-q^{4n})^c$$

and

$$V := \sum_{n=1}^{\infty}\left\{x\sigma_3(n) + y\sigma_3(\frac{n}{2}) + z\sigma_3(\frac{n}{4})\right\}q^n.$$

We list two following tables:

Table 3. Coefficients of q^n in U and V for n $(8 \leq n \leq 15)$ when $(a, b, c, x, y, z) = (-8, 16, 0, 1, -1, 0)$

n	8	9	10	11	12	13	14	15
U	512	757	1008	1332	1792	2198	2752	3528
V	512	757	1008	1332	1792	2198	2752	3528

Table 4. Coefficients of q^n in U and V for n $(8 \leq n \leq 15)$ when $(a, b, c, x, y, z) = (8, -8, 8, 1, -17, 16)$

n	8	9	10	11	12	13	14	15
U	-512	757	-1008	1332	-1792	2198	-2752	3528
V	-512	757	-1008	1332	-1792	2198	-2752	3528

□

Remark 2.3. In a similar manner to Remark 2.1 we can find rational numbers α, β, and γ such that

$$\alpha M(q) + \beta M(q^2) + \gamma M(q^4) = x^2 z^4.$$

By (1.3), (1.4), and (1.5) the above identity is written as

$$\begin{aligned}
&\alpha M(q) + \beta M(q^2) + \gamma M(q^4) \\
&= (\alpha + \beta + \gamma)\, z^4 + (14\alpha - \beta - \gamma)\, xz^4 + \left(\alpha + \beta + \frac{1}{16}\gamma\right) x^2 z^4 \\
&= x^2 z^4
\end{aligned}$$

and so

$$\alpha + \beta + \gamma = 0, \quad 14\alpha - \beta - \gamma = 0, \quad \alpha + \beta + \frac{1}{16}\gamma = 1$$

thus we have

$$\alpha = 0, \quad \beta = \frac{16}{15}, \quad \gamma = -\frac{16}{15}.$$

Therefore we can show that

$$\frac{16}{15} M(q^2) - \frac{16}{15} M(q^4) = x^2 z^4. \tag{2.30}$$

From (1.6), (2.1), and (2.30) we can deduce that

$$\begin{aligned}
&\frac{16}{15} M(q^2) - \frac{16}{15} M(q^4) \\
&= x^2 z^4 \\
&= (xz^2)^2 \\
&= \left\{ 16q \prod_{n=1}^{\infty} \frac{(1-q^{4n})^8}{(1-q^{2n})^4} \right\}^2 \\
&= 256 q^2 \prod_{n=1}^{\infty} (1-q^{2n})^{-8} (1-q^{4n})^{16} \\
&= \frac{16}{15}\left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^{2n}\right) - \frac{16}{15}\left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^{4n}\right) \\
&= 256 \sum_{n=1}^{\infty} \sigma_3(n) q^{2n} - 256 \sum_{n=1}^{\infty} \sigma_3(n) q^{4n}
\end{aligned}$$

which summarizes that

$$q^2 \prod_{n=1}^{\infty} (1-q^{2n})^{-8} (1-q^{4n})^{16} = \sum_{n=1}^{\infty} \left\{ \sigma_3\left(\frac{n}{2}\right) - \sigma_3\left(\frac{n}{4}\right) \right\} q^n. \tag{2.31}$$

Lastly we try to obtain another similar form as (2.31) but we can see that (2.31) is the only solution in Theorem 2.3.

Theorem 2.3. *If a, b, c, x, y, z are integers such that*

$$q^2 \prod_{n=1}^{\infty}(1-q^n)^a(1-q^{2n})^b(1-q^{4n})^c = \sum_{n=1}^{\infty}\left\{x\sigma_3(n)+y\sigma_3(\frac{n}{2})+z\sigma_3(\frac{n}{4})\right\}q^n \tag{2.32}$$

holds, then

$$(a,b,c,x,y,z)=(0,-8,16,0,1,-1).$$

Proof. Using MAPLE, we find that the left hand side of (2.32) is

$$\begin{aligned}
&q^2 \prod_{n=1}^{\infty}(1-q^n)^a(1-q^{2n})^b(1-q^{4n})^c \\
&= q^2 - aq^3 + \left(\frac{1}{2}a^2 - \frac{3}{2}a - b\right)q^4 + \left(-\frac{1}{6}a^3 + \frac{3}{2}a^2 - \frac{4}{3}a + ab\right)q^5 \\
&\quad + \left(\frac{1}{24}a^4 - \frac{3}{4}a^3 + \frac{59}{24}a^2 - \frac{7}{4}a - \frac{1}{2}a^2b + \frac{3}{2}ab + \frac{1}{2}b^2 - \frac{3}{2}b - c\right)q^6 \\
&\quad + \left(-\frac{1}{120}a^5 + \frac{1}{4}a^4 - \frac{43}{24}a^3 + \frac{15}{4}a^2 - \frac{6}{5}a + \frac{1}{6}a^3b - \frac{3}{2}a^2b + \frac{17}{6}ab - \frac{1}{2}ab^2 + ac\right)q^7 \\
&\quad + \left(\frac{1}{720}a^6 - \frac{1}{16}a^5 + \frac{113}{144}a^4 - \frac{55}{16}a^3 + \frac{1697}{360}a^2 - 2a - \frac{1}{24}a^4b + \frac{3}{4}a^3b - \frac{77}{24}a^2b \right.\\
&\qquad \left. +4ab + \frac{1}{4}a^2b^2 - \frac{3}{4}ab^2 - \frac{1}{6}b^3 + \frac{3}{2}b^2 - \frac{4}{3}b - \frac{1}{2}a^2c + \frac{3}{2}ac + bc\right)q^8 + \cdots .
\end{aligned}$$

On the other hand the right hand side of (2.32) is

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left\{x\sigma_3(n)+y\sigma_3(\frac{n}{2})+z\sigma_3(\frac{n}{4})\right\}q^n \\
&= xq + (9x+y)\,q^2 + 28xq^3 + (73x+9y+z)\,q^4 + 126xq^5 + (252x+28y)\,q^6 \\
&\quad + 344xq^7 + (585x+73y+9z)\,q^8 + \cdots .
\end{aligned}$$

Equating coefficients of (2.32), we obtain that

$$0 = x, \tag{2.33}$$

$$1 = 9x + y, \tag{2.34}$$

$$-a = 28x, \tag{2.35}$$

$$\frac{1}{2}a^2 - \frac{3}{2}a - b = 73x + 9y + z, \tag{2.36}$$

$$-\frac{1}{6}a^3 + \frac{3}{2}a^2 - \frac{4}{3}a + ab = 126x, \tag{2.37}$$

$$\frac{1}{24}a^4 - \frac{3}{4}a^3 + \frac{59}{24}a^2 - \frac{7}{4}a - \frac{1}{2}a^2b + \frac{3}{2}ab + \frac{1}{2}b^2 - \frac{3}{2}b - c = 252x + 28y, \tag{2.38}$$

$$-\frac{1}{120}a^5+\frac{1}{4}a^4-\frac{43}{24}a^3+\frac{15}{4}a^2-\frac{6}{5}a+\frac{1}{6}a^3b-\frac{3}{2}a^2b+\frac{17}{6}ab-\frac{1}{2}ab^2+ac=344x, \tag{2.39}$$

and

$$\begin{aligned}&\frac{1}{720}a^6-\frac{1}{16}a^5+\frac{113}{144}a^4-\frac{55}{16}a^3+\frac{1697}{360}a^2-2a-\frac{1}{24}a^4b+\frac{3}{4}a^3b-\frac{77}{24}a^2b\\&+4ab+\frac{1}{4}a^2b^2-\frac{3}{4}ab^2-\frac{1}{6}b^3+\frac{3}{2}b^2-\frac{4}{3}b-\frac{1}{2}a^2c+\frac{3}{2}ac+bc=585x+73y+9z.\end{aligned} \tag{2.40}$$

From (2.33), (2.34), and (2.35) we have

$$a=0 \quad \text{and} \quad y=1. \tag{2.41}$$

Also by (2.36), (2.38), and (2.40), we obtain

$$b=-8, \quad c=16, \quad z=-1. \tag{2.42}$$

To ensure the proof we compare some coefficients in

$$W:=q^2\prod_{n=1}^{\infty}(1-q^n)^a(1-q^{2n})^b(1-q^{4n})^c$$

and

$$Z:=\sum_{n=1}^{\infty}\left\{x\sigma_3(n)+y\sigma_3(\frac{n}{2})+z\sigma_3(\frac{n}{4})\right\}q^n.$$

We show the following table :

Table 5. Coefficients of q^n in W and Z for n ($9 \le n \le 16$) when $(a,b,c,x,y,z)=(0,-8,16,0,1,-1)$

n	9	10	11	12	13	14	15	16
W	0	126	0	224	0	344	0	512
Z	0	126	0	224	0	344	0	512

□

Theorem 2.4. *There are no integers r, a, b, c, x, y, z with $r \ge 3$ such that*

$$q^r\prod_{n=1}^{\infty}(1-q^n)^a(1-q^{2n})^b(1-q^{4n})^c=\sum_{n=1}^{\infty}\left\{x\sigma_3(n)+y\sigma_3(\frac{n}{2})+z\sigma_3(\frac{n}{4})\right\}q^n. \tag{2.43}$$

Proof. Assume that there exist integers r, a, b, c, x, y, and z with $r \geq 3$ such that (2.43) is satisfied. Then we obtain

$$\begin{aligned}
&q^r \prod_{n=1}^{\infty}(1-q^n)^a(1-q^{2n})^b(1-q^{4n})^c \\
&= q^r \left\{1 - aq + \left(\frac{1}{2}a^2 - \frac{3}{2}a - b\right)q^2 + \left(-\frac{1}{6}a^3 + \frac{3}{2}a^2 - \frac{4}{3}a + ab\right)q^3 \right.\\
&\quad + \left(\frac{1}{24}a^4 - \frac{3}{4}a^3 + \frac{59}{24}a^2 - \frac{7}{4}a - \frac{1}{2}a^2b + \frac{3}{2}ab + \frac{1}{2}b^2 - \frac{3}{2}b - c\right)q^4 \\
&\quad + \left(-\frac{1}{120}a^5 + \frac{1}{4}a^4 - \frac{43}{24}a^3 + \frac{15}{4}a^2 - \frac{6}{5}a + \frac{1}{6}a^3b - \frac{3}{2}a^2b + \frac{17}{6}ab \right.\\
&\qquad \left.\left. - \frac{1}{2}ab^2 + ac\right)q^5 + \cdots \right\} \\
&= q^r - aq^{r+1} + \left(\frac{1}{2}a^2 - \frac{3}{2}a - b\right)q^{r+2} + \left(-\frac{1}{6}a^3 + \frac{3}{2}a^2 - \frac{4}{3}a + ab\right)q^{r+3} \\
&\quad + \left(\frac{1}{24}a^4 - \frac{3}{4}a^3 + \frac{59}{24}a^2 - \frac{7}{4}a - \frac{1}{2}a^2b + \frac{3}{2}ab + \frac{1}{2}b^2 - \frac{3}{2}b - c\right)q^{r+4} \\
&\quad + \left(-\frac{1}{120}a^5 + \frac{1}{4}a^4 - \frac{43}{24}a^3 + \frac{15}{4}a^2 - \frac{6}{5}a + \frac{1}{6}a^3b - \frac{3}{2}a^2b + \frac{17}{6}ab \right.\\
&\qquad \left. - \frac{1}{2}ab^2 + ac\right)q^{r+5} + \cdots \\
&= \sum_{n=1}^{\infty}\left\{x\sigma_3(n) + y\sigma_3(\frac{n}{2}) + z\sigma_3(\frac{n}{4})\right\}q^n \\
&= xq + (9x+y)\,q^2 + 28xq^3 + (73x+9y+z)\,q^4 + 126xq^5 + (252x+28y)\,q^6 + \cdots
\end{aligned}$$

which summarizes that

$$\begin{aligned}
&q^r - aq^{r+1} + \left(\frac{1}{2}a^2 - \frac{3}{2}a - b\right)q^{r+2} + \left(-\frac{1}{6}a^3 + \frac{3}{2}a^2 - \frac{4}{3}a + ab\right)q^{r+3} \\
&\quad + \left(\frac{1}{24}a^4 - \frac{3}{4}a^3 + \frac{59}{24}a^2 - \frac{7}{4}a - \frac{1}{2}a^2b + \frac{3}{2}ab + \frac{1}{2}b^2 - \frac{3}{2}b - c\right)q^{r+4} \\
&\quad + \left(-\frac{1}{120}a^5 + \frac{1}{4}a^4 - \frac{43}{24}a^3 + \frac{15}{4}a^2 - \frac{6}{5}a + \frac{1}{6}a^3b - \frac{3}{2}a^2b + \frac{17}{6}ab \right.\\
&\qquad \left. - \frac{1}{2}ab^2 + ac\right)q^{r+5} + \cdots \\
&= xq + (9x+y)\,q^2 + 28xq^3 + (73x+9y+z)\,q^4 + 126xq^5 + (252x+28y)\,q^6 + \cdots .
\end{aligned} \tag{2.44}$$

If $r \geq 5$, then Eq. (2.44) should satisfy that

$$x = 9x + y = 28x = 73x + 9y + z = 0$$

so that

$$x = 0 = y = z.$$

This shows that the right hand side of (2.43) is equal to zero, which contradicts that the left hand side of (2.43) has $q^r \neq 0$. Therefore we only consider the cases $r = 3$ and $r = 4$.

If $r = 3$, then (2.44) becomes

$$\begin{aligned} & q^3 - aq^4 + \left(\frac{1}{2}a^2 - \frac{3}{2}a - b\right) q^5 + \left(-\frac{1}{6}a^3 + \frac{3}{2}a^2 - \frac{4}{3}a + ab\right) q^6 \\ & \quad + \left(\frac{1}{24}a^4 - \frac{3}{4}a^3 + \frac{59}{24}a^2 - \frac{7}{4}a - \frac{1}{2}a^2 b + \frac{3}{2}ab + \frac{1}{2}b^2 - \frac{3}{2}b - c\right) q^7 \\ & \quad + \left(-\frac{1}{120}a^5 + \frac{1}{4}a^4 - \frac{43}{24}a^3 + \frac{15}{4}a^2 - \frac{6}{5}a + \frac{1}{6}a^3 b - \frac{3}{2}a^2 b + \frac{17}{6}ab \right. \\ & \qquad \left. - \frac{1}{2}ab^2 + ac\right) q^8 + \cdots \\ & = xq + (9x + y)\, q^2 + 28xq^3 + (73x + 9y + z)\, q^4 + 126xq^5 + (252x + 28y)\, q^6 + \cdots \end{aligned}$$

so that

$$x = 0, \qquad 9x + y = 0, \qquad \text{and} \qquad 28x = 1,$$

which is a contradiction.

If $r = 4$, then (2.44) can be written as

$$\begin{aligned} & q^4 - aq^5 + \left(\frac{1}{2}a^2 - \frac{3}{2}a - b\right) q^6 + \left(-\frac{1}{6}a^3 + \frac{3}{2}a^2 - \frac{4}{3}a + ab\right) q^7 \\ & \quad + \left(\frac{1}{24}a^4 - \frac{3}{4}a^3 + \frac{59}{24}a^2 - \frac{7}{4}a - \frac{1}{2}a^2 b + \frac{3}{2}ab + \frac{1}{2}b^2 - \frac{3}{2}b - c\right) q^8 \\ & \quad + \left(-\frac{1}{120}a^5 + \frac{1}{4}a^4 - \frac{43}{24}a^3 + \frac{15}{4}a^2 - \frac{6}{5}a + \frac{1}{6}a^3 b - \frac{3}{2}a^2 b + \frac{17}{6}ab \right. \\ & \qquad \left. - \frac{1}{2}ab^2 + ac\right) q^9 + \cdots \\ & = xq + (9x + y)\, q^2 + 28xq^3 + (73x + 9y + z)\, q^4 + 126xq^5 + (252x + 28y)\, q^6 + \cdots \end{aligned}$$

thus

$$x = 0, \quad 9x + y = 0, \quad 28x = 0, \quad \text{and} \quad 73x + 9y + z = 1$$

which implies that

$$x = 0, \quad y = 0, \quad z = 1, \quad -a = 0, \quad \text{and} \quad \frac{1}{2}a^2 - \frac{3}{2}a - b = 0.$$

Then, (2.43) takes the form

$$q^4 \prod_{n=1}^{\infty} (1 - q^{4n})^c = \sum_{n=1}^{\infty} \sigma_3(\frac{n}{4}) q^n = \sum_{n=1}^{\infty} \sigma_3(n) q^{4n}$$

and so replacing q^4 by q we deduce that

$$q \prod_{n=1}^{\infty} (1 - q^n)^c = \sum_{n=1}^{\infty} \sigma_3(n) q^n,$$

which does not occur by Theorem 2.2. Therefore we complete the proof. □

Proof of Theorem 1.1. The case $(0, 1, 0, 0, 0, 0, 0, 0)$ is obvious directly by inserting each values into (1.7). And the other cases is proved easily by Theorem 2.1, Theorem 2.2, Theorem 2.3, and Theorem 2.4. □

3 Conclusion

We construct a product-to-sum as follows

$$q^r \prod_{n=1}^{\infty}(1-q^n)^a(1-q^{2n})^b(1-q^{4n})^c = u + \sum_{n=1}^{\infty}\left\{x\sigma_3(n) + y\sigma_3(\frac{n}{2}) + z\sigma_3(\frac{n}{4})\right\} q^n$$

then we find the integers r, u, a, b, c, x, y, and z with $r \geq 0$.

Competing Interests

Author has declared that no competing interests exist.

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