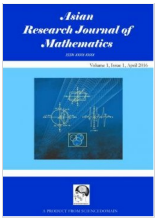

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Analytical Study of a System of Difference Equation

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

We study the qualitative behavior of a predator-prey model, where the carrying capacity of the predators environment is proportional to the number of prey. The considered system is given by the following rational difference equations:

$$x_{n+1} = \frac{x_n y_{n-2}}{y_{n-1} + y_{n-2}}, \quad y_{n+1} = \frac{y_n x_{n-2}}{\pm x_{n-1} \pm x_{n-2}}, \quad n = 0, 1, \cdots,$$

where the initial conditions $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0$ are arbitrary positive real numbers. Also, we give specific form of the solutions of some special cases of this equation. Some numerical examples are given to verify our theoretical results.

Keywords: Difference equations; Recursive sequences; Stability; Boundedness.

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1 Introduction

Difference equations appear as natural descriptions of observed evolution phenomena because most measurements of time evolving variables are discrete and as such these equations are in their own right important mathematical models. More importantly, difference equations also appear in the study of discretization methods for differential equations. Several results in the theory of difference equations have been obtained as more or less natural discrete analogues of corresponding results of differential equations. The study of rational difference equations of order greater than one is quite challenging and rewarding because some prototypes for the development of the basic theory of the global behavior of nonlinear difference equations of order greater than one come from the results for rational difference equations. However, there have not been any effective general methods to deal with the global behavior of rational difference equations of order greater than one so far. Therefore, the study of rational difference equations of order greater than one is worth further consideration see [1, 2, 3, 4, 5, 6].

The study and solution of nonlinear rational recursive sequence of high order is quite challenging and rewarding. This can be attributed to the fact that many real life phenomena are modeled using difference equations. Some economical and biological examples can be seen in [1, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 2, 3, 4, 5, 6]. It is commonly known that nonlinear difference equations are able to produce and present sophisticated behaviors regardless their orders. Recently, there has been a lot of interest in studying the qualitative properties of rational recursive sequences. Furthermore diverse nonlinear trend occurring in science and engineering can be modeled by such equations and the solution about such equations offer prototypes towards the development of the theory. However, there have not been any suitable general method to deal with the global behavior of rational difference equations of high order so far. Therefore, the study of rational difference equations of order greater than one is worth further consideration.

A great example of both facts are Ricatti difference equations since the plenty of the dynamics of Ricatti equations is very well-known, and a particular case of these equations provides the classical Beverton-Holt model on the dynamics of exploited fish populations.

There are many articles on the difference equations systems [18, 19, 20]. For example, the periodicity the positive solutions of the rational difference equations systems

$$x_{n+1} = \frac{m}{y_n}, \quad y_{n+1} = \frac{py_n}{x_{n-1}y_{n-1}}$$

has been obtained by Cinar[19] .

Din et al[1]. studied the equilibrium points, local and global stability and periodicity of positive solutions of a fourth-order system of rational difference equations of the form

$$x_{n+1} = \frac{\alpha x_{n-3}}{\beta + \gamma y_n y_{n-1} y_{n-2} y_{n-3}}, \quad y_{n+1} = \frac{\alpha_1 y_{n-3}}{\beta_1 + \gamma_1 x_n x_{n-1} x_{n-2} x_{n-3}}$$

Similarly, a large number of other difference equations and nonlinear systems of the rational difference equations were studied.

Ahmed [21] investigated the global behavior for the system of rational difference equations

$$x_{n+1} = \frac{x_{n-1}}{a + y_n x_{n-1}}, \quad y_{n+1} = \frac{y_{n-1}}{b + x_n y_{n-1}}$$

Bao [22] concerned with dynamics of the solution to the system of two second-order nonlinear difference equations

$$x_{n+1} = A + \frac{x_n}{x_{n-1}y_{n-1}}, \quad y_{n+1} = A + \frac{y_n}{x_{n-1}y_{n-1}}$$

Din [23] investigated the dynamics of the following system of fourth-order rational difference equations

$$x_{n+1} = \frac{\alpha_1 x_{n-3}}{\beta_1 y_n y_{n-1} x_{n-2} x_{n-3}}, \quad y_{n+1} = \frac{\alpha_2 y_{n-3}}{\beta_2 x_n x_{n-1} y_{n-2} y_{n-3}}$$

Our aim in this paper, is to investigate the behavior of the solution of the following nonlinear difference equation

$$x_{n+1} = \frac{x_n y_{n-2}}{y_{n-1} + y_{n-2}}, \quad y_{n+1} = \frac{y_n x_{n-2}}{\pm x_{n-1} \pm x_{n-2}}, \quad n = 0, 1, ..., \tag{1.1}$$

where the initial conditions $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0$ are arbitrary positive real numbers.
We find a specific form of the solutions of some special cases of Eq. (1.1) and give numerical examples of each case.

2 First Case : On the Difference Equation $x_{n+1} = \frac{x_n y_{n-2}}{y_{n-1} + y_{n-2}}, y_{n+1} = \frac{y_n x_{n-2}}{x_{n-1} + x_{n-2}}$

In this section we study the following special case of Eq. (1.1):

$$x_{n+1} = \frac{x_n y_{n-2}}{y_{n-1} + y_{n-2}}, \quad y_{n+1} = \frac{y_n x_{n-2}}{x_{n-1} + x_{n-2}}, \quad n = 0, 1, ..., \tag{2.1}$$

where the initial conditions $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0$ are arbitrary positive real numbers.

Theorem 1. Let $\{x_n, y_n\}_{n=-2}^{\infty}$ be a solution of Eq. (2.1). Then for $n = 0, 1, ...,$

$$x_{4n-2} = c\prod_{i=0}^{n-1}\frac{(af_{2i-2}+bf_{2i-1})(bf_{2i-2}+cf_{2i-1})(df_{2i-1}+ef_{2i})(ef_{2i-1}+gf_{2i})}{(af_{2i-1}+bf_{2i})(bf_{2i-1}+cf_{2i})(df_{2i}+ef_{2i+1})(ef_{2i}+gf_{2i+1})},$$

$$x_{4n-1} = b\prod_{i=0}^{n-1}\frac{(af_{2i-2}+bf_{2i-1})(bf_{2i}+cf_{2i+1})(df_{2i-1}+ef_{2i})(ef_{2i-1}+gf_{2i})}{(af_{2i-1}+bf_{2i})(bf_{2i+1}+cf_{2i+2})(df_{2i}+ef_{2i+1})(ef_{2i}+gf_{2i+1})},$$

$$x_{4n} = a\prod_{i=0}^{n-1}\frac{(af_{2i}+bf_{2i+1})(bf_{2i}+cf_{2i+1})(df_{2i-1}+ef_{2i})(ef_{2i-1}+gf_{2i})}{(af_{2i+1}+bf_{2i+2})(bf_{2i+1}+cf_{2i+2})(df_{2i}+ef_{2i+1})(ef_{2i}+gf_{2i+1})},$$

$$x_{4n+1} = \frac{ag}{(e+g)}\prod_{i=0}^{n-1}\frac{(af_{2i}+bf_{2i+1})(bf_{2i}+cf_{2i+1})(df_{2i-1}+ef_{2i})(ef_{2i+1}+gf_{2i+2})}{(af_{2i+1}+bf_{2i+2})(bf_{2i+1}+cf_{2i+2})(df_{2i}+ef_{2i+1})(ef_{2i+2}+gf_{2i+3})},$$

$$y_{4n-2} = g\prod_{i=0}^{n-1}\frac{(af_{2i-1}+bf_{2i})(bf_{2i-1}+cf_{2i})(df_{2i-2}+ef_{2i-1})(ef_{2i-2}+gf_{2i-1})}{(af_{2i}+bf_{2i+1})(bf_{2i}+cf_{2i+1})(df_{2i-1}+ef_{2i})(ef_{2i-1}+gf_{2i})},$$

$$y_{4n-1} = e\prod_{i=0}^{n-1}\frac{(af_{2i-1}+bf_{2i})(bf_{2i-1}+cf_{2i})(df_{2i-2}+ef_{2i-1})(ef_{2i}+gf_{2i+1})}{(af_{2i}+bf_{2i+1})(bf_{2i}+cf_{2i+1})(df_{2i-1}+ef_{2i})(ef_{2i+1}+gf_{2i+2})},$$

$$y_{4n} = d\prod_{i=0}^{n-1}\frac{(af_{2i-1}+bf_{2i})(bf_{2i-1}+cf_{2i})(df_{2i}+ef_{2i+1})(ef_{2i}+gf_{2i+1})}{(af_{2i}+bf_{2i+1})(bf_{2i}+cf_{2i+1})(df_{2i+1}+ef_{2i+2})(ef_{2i+1}+gf_{2i+2})},$$

$$y_{4n+1} = \frac{dc}{(b+c)}\prod_{i=0}^{n-1}\frac{(af_{2i-1}+bf_{2i})(bf_{2i+1}+cf_{2i+2})(df_{2i}+ef_{2i+1})(ef_{2i}+gf_{2i+1})}{(af_{2i}+bf_{2i+1})(bf_{2i+2}+cf_{2i+3})(df_{2i+1}+ef_{2i+2})(ef_{2i+1}+gf_{2i+2})},$$

where $x_{-2} = c, x_{-1} = b, x_0 = a, y_{-2} = g, y_{-1} = e, y_0 = d$ and $\{f_m\}_{m=-2}^{\infty} = \{1, 0, 1, 1, 2, 3, 5, 8, 13, ...\}$.

Proof: We use an inductive proof for this rational recursive sequences. It is easy to see that for $n = 0$, the result holds. Suppose that $n > 0$ and that the assumption is satisfied for $n - 1$. That is;

$$x_{4n-6} = c\prod_{i=0}^{n-2}\frac{(af_{2i-2}+bf_{2i-1})(bf_{2i-2}+cf_{2i-1})(df_{2i-1}+ef_{2i})(ef_{2i-1}+gf_{2i})}{(af_{2i-1}+bf_{2i})(bf_{2i-1}+cf_{2i})(df_{2i}+ef_{2i+1})(ef_{2i}+gf_{2i+1})},$$

$$x_{4n-5} = b\prod_{i=0}^{n-2}\frac{(af_{2i-2}+bf_{2i-1})(bf_{2i}+cf_{2i+1})(df_{2i-1}+ef_{2i})(ef_{2i-1}+gf_{2i})}{(af_{2i-1}+bf_{2i})(bf_{2i+1}+cf_{2i+2})(df_{2i}+ef_{2i+1})(ef_{2i}+gf_{2i+1})},$$

$$x_{4n-4} = a\prod_{i=0}^{n-2}\frac{(af_{2i}+bf_{2i+1})(bf_{2i}+cf_{2i+1})(df_{2i-1}+ef_{2i})(ef_{2i-1}+gf_{2i})}{(af_{2i+1}+bf_{2i+2})(bf_{2i+1}+cf_{2i+2})(df_{2i}+ef_{2i+1})(ef_{2i}+gf_{2i+1})},$$

$$x_{4n-3} = \frac{ag}{(e+g)}\prod_{i=0}^{n-2}\frac{(af_{2i}+bf_{2i+1})(bf_{2i}+cf_{2i+1})(df_{2i-1}+ef_{2i})(ef_{2i+1}+gf_{2i+2})}{(af_{2i+1}+bf_{2i+2})(bf_{2i+1}+cf_{2i+2})(df_{2i}+ef_{2i+1})(ef_{2i+2}+gf_{2i+3})},$$

$$y_{4n-6} = g\prod_{i=0}^{n-2}\frac{(af_{2i-1}+bf_{2i})(bf_{2i-1}+cf_{2i})(df_{2i-2}+ef_{2i-1})(ef_{2i-2}+gf_{2i-1})}{(af_{2i}+bf_{2i+1})(bf_{2i}+cf_{2i+1})(df_{2i-1}+ef_{2i})(ef_{2i-1}+gf_{2i})},$$

$$y_{4n-5} = e\prod_{i=0}^{n-2}\frac{(af_{2i-1}+bf_{2i})(bf_{2i-1}+cf_{2i})(df_{2i-2}+ef_{2i-1})(ef_{2i}+gf_{2i+1})}{(af_{2i}+bf_{2i+1})(bf_{2i}+cf_{2i+1})(df_{2i-1}+ef_{2i})(ef_{2i+1}+gf_{2i+2})},$$

$$y_{4n-4} = d\prod_{i=0}^{n-2}\frac{(af_{2i-1}+bf_{2i})(bf_{2i-1}+cf_{2i})(df_{2i}+ef_{2i+1})(ef_{2i}+gf_{2i+1})}{(af_{2i}+bf_{2i+1})(bf_{2i}+cf_{2i+1})(df_{2i+1}+ef_{2i+2})(ef_{2i+1}+gf_{2i+2})},$$

$$y_{4n-3} = \frac{dc}{(b+c)}\prod_{i=0}^{n-2}\frac{(af_{2i-1}+bf_{2i})(bf_{2i+1}+cf_{2i+2})(df_{2i}+ef_{2i+1})(ef_{2i}+gf_{2i+1})}{(af_{2i}+bf_{2i+1})(bf_{2i+2}+cf_{2i+3})(df_{2i+1}+ef_{2i+2})(ef_{2i+1}+gf_{2i+2})},$$

Now we find from Eq. (2.1) that

$$x_{4n-2} = \frac{x_{4n-3}y_{4n-5}}{y_{4n-4}+y_{4n-5}}$$

$$= \frac{ag}{(e+g)}\frac{\prod_{i=0}^{n-2}\frac{(af_{2i}+bf_{2i+1})(bf_{2i}+cf_{2i+1})(df_{2i-1}+ef_{2i})(ef_{2i+1}+gf_{2i+2})}{(af_{2i+1}+bf_{2i+2})(bf_{2i+1}+cf_{2i+2})(df_{2i}+ef_{2i+1})(ef_{2i+2}+gf_{2i+3})}e\prod_{i=0}^{n-2}\frac{(af_{2i-1}+bf_{2i})(bf_{2i-1}+cf_{2i})(df_{2i-2}+ef_{2i-1})(ef_{2i}+gf_{2i+1})}{(af_{2i}+bf_{2i+1})(bf_{2i}+cf_{2i+1})(df_{2i-1}+ef_{2i})(ef_{2i+1}+gf_{2i+2})}}{d\prod_{i=0}^{n-2}\frac{(af_{2i-1}+bf_{2i})(bf_{2i-1}+cf_{2i})(df_{2i}+ef_{2i+1})(ef_{2i}+gf_{2i+1})}{(af_{2i}+bf_{2i+1})(bf_{2i}+cf_{2i+1})(df_{2i+1}+ef_{2i+2})(ef_{2i+1}+gf_{2i+2})}+e\prod_{i=0}^{n-2}\frac{(af_{2i-1}+bf_{2i})(bf_{2i-1}+cf_{2i})(df_{2i-2}+ef_{2i-1})(ef_{2i}+gf_{2i+1})}{(af_{2i}+bf_{2i+1})(bf_{2i}+cf_{2i+1})(df_{2i-1}+ef_{2i})(ef_{2i+1}+gf_{2i+2})}}$$

$$= \frac{aeg}{(e+g)}\prod_{i=0}^{n-2}\frac{\frac{(af_{2i}+bf_{2i+1})(bf_{2i}+cf_{2i+1})(df_{2i-1}+ef_{2i})(ef_{2i+1}+gf_{2i+2})}{(af_{2i+1}+bf_{2i+2})(bf_{2i+1}+cf_{2i+2})(df_{2i}+ef_{2i+1})(ef_{2i+2}+gf_{2i+3})}\frac{(af_{2i-1}+bf_{2i})(bf_{2i-1}+cf_{2i})(df_{2i-2}+ef_{2i-1})(ef_{2i}+gf_{2i+1})}{(af_{2i}+bf_{2i+1})(bf_{2i}+cf_{2i+1})(df_{2i-1}+ef_{2i})(ef_{2i+1}+gf_{2i+2})}}{d\frac{(af_{2i-1}+bf_{2i})(bf_{2i-1}+cf_{2i})(df_{2i}+ef_{2i+1})(ef_{2i}+gf_{2i+1})}{(af_{2i}+bf_{2i+1})(bf_{2i}+cf_{2i+1})(df_{2i+1}+ef_{2i+2})(ef_{2i+1}+gf_{2i+2})}+e\frac{(af_{2i-1}+bf_{2i})(bf_{2i-1}+cf_{2i})(df_{2i-2}+ef_{2i-1})(ef_{2i}+gf_{2i+1})}{(af_{2i}+bf_{2i+1})(bf_{2i}+cf_{2i+1})(df_{2i-1}+ef_{2i})(ef_{2i+1}+gf_{2i+2})}}$$

$$= \frac{aeg}{(e+g)}\prod_{i=0}^{n-2}\frac{\frac{(af_{2i-1}+bf_{2i})(bf_{2i-1}+cf_{2i})(df_{2i-2}+ef_{2i-1})(ef_{2i}+gf_{2i+1})}{(af_{2i+1}+bf_{2i+2})(bf_{2i+1}+cf_{2i+2})(df_{2i}+ef_{2i+1})(ef_{2i+2}+gf_{2i+3})}}{\frac{(af_{2i-1}+bf_{2i})(bf_{2i-1}+cf_{2i})(ef_{2i}+gf_{2i+1})}{(af_{2i}+bf_{2i+1})(bf_{2i}+cf_{2i+1})(ef_{2i+1}+gf_{2i+2})}\left(d\frac{(df_{2i}+ef_{2i+1})}{(df_{2i+1}+ef_{2i+2})}+e\frac{(df_{2i-2}+ef_{2i-1})}{(df_{2i-1}+ef_{2i})}\right)}$$

$$= \frac{aeg}{(e+g)}\prod_{i=0}^{n-2}\frac{\frac{(df_{2i-2}+ef_{2i-1})}{(af_{2i+1}+bf_{2i+2})(bf_{2i+1}+cf_{2i+2})(df_{2i}+ef_{2i+1})(ef_{2i+2}+gf_{2i+3})}}{\frac{1}{(af_{2i}+bf_{2i+1})(bf_{2i}+cf_{2i+1})(ef_{2i+1}+gf_{2i+2})}\left(d\frac{(df_{2i}+ef_{2i+1})}{(df_{2i+1}+ef_{2i+2})}+e\frac{(df_{2i-2}+ef_{2i-1})}{(df_{2i-1}+ef_{2i})}\right)}$$

$$= \frac{aeg}{(e+g)}\prod_{i=0}^{n-2}\frac{\frac{(af_{2i}+bf_{2i+1})(bf_{2i}+cf_{2i+1})(df_{2i-2}+ef_{2i-1})(ef_{2i+1}+gf_{2i+2})}{(af_{2i+1}+bf_{2i+2})(bf_{2i+1}+cf_{2i+2})(df_{2i}+ef_{2i+1})(ef_{2i+2}+gf_{2i+3})}}{d\frac{(df_{2i}+ef_{2i+1})}{(df_{2i+1}+ef_{2i+2})}+e\frac{(df_{2i-2}+ef_{2i-1})}{(df_{2i-1}+ef_{2i})}}$$

$$= \frac{aeg}{(e+g)} \frac{\prod_{i=0}^{n-2} \frac{(af_{2i}+bf_{2i+1})(bf_{2i}+cf_{2i+1})(ef_{2i+1}+gf_{2i+2})}{(af_{2i+1}+bf_{2i+2})(bf_{2i+1}+cf_{2i+2})(ef_{2i+2}+gf_{2i+3})} \prod_{i=0}^{n-2} \frac{(df_{2i-2}+ef_{2i-1})}{(df_{2i}+ef_{2i+1})}}{d\prod_{i=0}^{n-2} \frac{(df_{2i}+ef_{2i+1})}{(df_{2i+1}+ef_{2i+2})} + e\prod_{i=0}^{n-2} \frac{(df_{2i-2}+ef_{2i-1})}{(df_{2i-1}+ef_{2i})}}$$

$$= \frac{aeg}{(e+g)} \frac{\prod_{i'=1}^{n-1} \frac{(af_{2i'-2}+bf_{2i'-1})(bf_{2i'-2}+cf_{2i'-1})(ef_{2i'-1}+gf_{2i'})}{(af_{2i'-1}+bf_{2i'})(bf_{2i'-1}+cf_{2i'})(ef_{2i'}+gf_{2i'+1})} \frac{d}{(df_{2n-4}+ef_{2n-3})}}{d\prod_{i'=1}^{n-1} \frac{(df_{2i'-2}+ef_{2i'-1})}{(df_{2i'-1}+ef_{2i'})} + e\prod_{i=0}^{n-2} \frac{(df_{2i-2}+ef_{2i-1})}{(df_{2i-1}+ef_{2i})}}$$

$$= \frac{aeg}{(e+g)} \frac{\prod_{i'=1}^{n-1} \frac{(af_{2i'-2}+bf_{2i'-1})(bf_{2i'-2}+cf_{2i'-1})(ef_{2i'-1}+gf_{2i'})}{(af_{2i'-1}+bf_{2i'})(bf_{2i'-1}+cf_{2i'})(ef_{2i'}+gf_{2i'+1})} \frac{d}{(df_{2n-4}+ef_{2n-3})}}{e\prod_{i'=0}^{n-1} \frac{(df_{2i'-2}+ef_{2i'-1})}{(df_{2i'-1}+ef_{2i'})} + e\prod_{i=0}^{n-2} \frac{(df_{2i-2}+ef_{2i-1})}{(df_{2i-1}+ef_{2i})}}$$

$$= \frac{daeg}{(e+g)(df_{2n-4}+ef_{2n-3})} \frac{\prod_{i'=1}^{n-1} \frac{(af_{2i'-2}+bf_{2i'-1})(bf_{2i'-2}+cf_{2i'-1})(ef_{2i'-1}+gf_{2i'})}{(af_{2i'-1}+bf_{2i'})(bf_{2i'-1}+cf_{2i'})(ef_{2i'}+gf_{2i'+1})}}{e\prod_{i=0}^{n-1} \frac{(df_{2i-2}+ef_{2i-1})}{(df_{2i-1}+ef_{2i})}(1+\frac{(df_{2n-3}+ef_{2n-2})}{(df_{2n-4}+ef_{2n-3})})}$$

$$= \frac{daeg}{(e+g)(df_{2n-4}+ef_{2n-3})} \frac{\prod_{i'=1}^{n-1} \frac{(af_{2i'-2}+bf_{2i'-1})(bf_{2i'-2}+cf_{2i'-1})(ef_{2i'-1}+gf_{2i'})}{(af_{2i'-1}+bf_{2i'})(bf_{2i'-1}+cf_{2i'})(ef_{2i'}+gf_{2i'+1})}}{e\prod_{i=0}^{n-1} \frac{(df_{2i-2}+ef_{2i-1})}{(df_{2i-1}+ef_{2i})} \frac{(d(f_{2n-4}+f_{2n-3})+e(f_{2n-3}+f_{2n-2}))}{(df_{2n-4}+ef_{2n-3})}}$$

$$= \frac{dag}{(e+g)} \frac{\prod_{i'=1}^{n-1} \frac{(af_{2i'-2}+bf_{2i'-1})(bf_{2i'-2}+cf_{2i'-1})(ef_{2i'-1}+gf_{2i'})}{(af_{2i'-1}+bf_{2i'})(bf_{2i'-1}+cf_{2i'})(ef_{2i'}+gf_{2i'+1})}}{\prod_{i=0}^{n-1} \frac{(df_{2i-2}+ef_{2i-1})}{(df_{2i-1}+ef_{2i})}(d(f_{2n-4}+f_{2n-3})+e(f_{2n-3}+f_{2n-2}))}$$

$$= \frac{dag}{(e+g)} \frac{\prod_{i'=1}^{n-1} \frac{(af_{2i'-2}+bf_{2i'-1})(bf_{2i'-2}+cf_{2i'-1})(ef_{2i'-1}+gf_{2i'})}{(af_{2i'-1}+bf_{2i'})(bf_{2i'-1}+cf_{2i'})(ef_{2i'}+gf_{2i'+1})}}{\prod_{i=0}^{n-1} \frac{(df_{2i-2}+ef_{2i-1})}{(df_{2i-1}+ef_{2i})}(df_{2n-2}+ef_{2n-1})}$$

$$= dc\frac{\prod_{i'=0}^{n-1}\frac{(af_{2i'-2}+bf_{2i'-1})(bf_{2i'-2}+cf_{2i'-1})(ef_{2i'-1}+gf_{2i'})}{(af_{2i'-1}+bf_{2i'})(bf_{2i'-1}+cf_{2i'})(ef_{2i'}+gf_{2i'+1})}}{\prod_{i=0}^{n-1}\frac{(df_{2i-2}+ef_{2i-1})}{(df_{2i-1}+ef_{2i})}(df_{2n-2}+ef_{2n-1})}$$

$$= \frac{dc}{(df_{2n-2}+ef_{2n-1})}\prod_{i'=0}^{n-1}\frac{(af_{2i'-2}+bf_{2i'-1})(bf_{2i'-2}+cf_{2i'-1})(ef_{2i'-1}+gf_{2i'})}{(af_{2i'-1}+bf_{2i'})(bf_{2i'-1}+cf_{2i'})(ef_{2i'}+gf_{2i'+1})}\prod_{i=0}^{n-1}\frac{(df_{2i-1}+ef_{2i})}{(df_{2i-2}+ef_{2i-1})}$$

$$= \frac{dc}{(df_{2n-2}+ef_{2n-1})}\prod_{i=0}^{n-1}\frac{(af_{2i-2}+bf_{2i-1})(bf_{2i-2}+cf_{2i-1})(df_{2i-1}+ef_{2i})(ef_{2i-1}+gf_{2i})}{(af_{2i-1}+bf_{2i})(bf_{2i-1}+cf_{2i})(ef_{2i}+gf_{2i+1})}\frac{1}{\prod_{i=0}^{n-1}(df_{2i-2}+ef_{2i-1})}$$

$$= \frac{dc}{(df_{2n-2}+ef_{2n-1})}\prod_{i=0}^{n-1}\frac{(af_{2i-2}+bf_{2i-1})(bf_{2i-2}+cf_{2i-1})(df_{2i-1}+ef_{2i})(ef_{2i-1}+gf_{2i})}{(af_{2i-1}+bf_{2i})(bf_{2i-1}+cf_{2i})(ef_{2i}+gf_{2i+1})}\frac{1}{\prod_{i=0}^{n-1}(df_{2i-2}+ef_{2i-1})}$$

$$= c\prod_{i=0}^{n-1}\frac{(af_{2i-2}+bf_{2i-1})(bf_{2i-2}+cf_{2i-1})(df_{2i-1}+ef_{2i})(ef_{2i-1}+gf_{2i})}{(af_{2i-1}+bf_{2i})(bf_{2i-1}+cf_{2i})(ef_{2i}+gf_{2i+1})}\frac{1}{\prod_{i=1}^{n}(df_{2i-2}+ef_{2i-1})}$$

$$= c\prod_{i=0}^{n-1}\frac{(af_{2i-2}+bf_{2i-1})(bf_{2i-2}+cf_{2i-1})(df_{2i-1}+ef_{2i})(ef_{2i-1}+gf_{2i})}{(af_{2i-1}+bf_{2i})(bf_{2i-1}+cf_{2i})(ef_{2i}+gf_{2i+1})}\frac{1}{\prod_{i=0}^{n-1}(df_{2i}+ef_{2i+1})}$$

$$= c\prod_{i=0}^{n-1}\frac{(af_{2i-2}+bf_{2i-1})(bf_{2i-2}+cf_{2i-1})(df_{2i-1}+ef_{2i})(ef_{2i-1}+gf_{2i})}{(af_{2i-1}+bf_{2i})(bf_{2i-1}+cf_{2i})(df_{2i}+ef_{2i+1})(ef_{2i}+gf_{2i+1})}$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed.

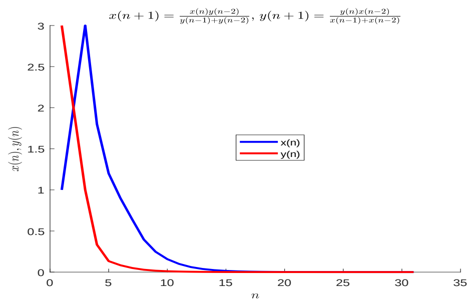

Fig. 1

Theorem 2. Let x_n, y_n be a positive solution of system (2.1), then every solution of system (2.1) is bounded and converges to zero.

Proof. It follows from Eq. (2.1) that

$$x_{n+1} = \frac{x_n y_{n-2}}{y_{n-1}+y_{n-2}} \leq x_n, \quad y_{n+1} = \frac{y_n x_{n-2}}{x_{n-1}+x_{n-2}} \leq y_n,$$

Then the sub-sequences $\{x_{2n-1}\}_{n=0}^{\infty}$, $\{x_{2n}\}_{n=0}^{\infty}$ are decreasing and so they are bounded above by $M_1 = \max\{x_{-1}, x_0\}$. Also, the sub-sequences $\{y_{2n-1}\}_{n=0}^{\infty}$, $\{y_{2n}\}_{n=0}^{\infty}$ are decreasing and so they are bounded above by $M_2 = \max\{y_{-1}, y_0\}$. The proof is complete.

For confirming the results of this section, we consider numerical example for $x_{-2} = 1, x_{-1} = 2, x_0 = 3, y_{-2} = 3, y_{-1} = 2, y_0 = 1$,(See Figure 1).

3 Second Case : On the Difference Equation $x_{n+1} = \dfrac{x_n y_{n-2}}{y_{n-1} + y_{n-2}}, \quad y_{n+1} = \dfrac{y_n x_{n-2}}{x_{n-1} - x_{n-2}}$

In this section we study the following special case of Eq. (1.1):

$$x_{n+1} = \frac{x_n y_{n-2}}{y_{n-1} + y_{n-2}}, \quad y_{n+1} = \frac{y_n x_{n-2}}{x_{n-1} - x_{n-2}}, \quad n = 0, 1, ..., \tag{3.1}$$

where the initial conditions $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0$ are arbitrary positive real numbers.

Theorem 3. Let $\{x_n, y_n\}_{n=-2}^{\infty}$ be a solution of Eq. (3.1). Then all solutions of system (3.1) are periodic with period twelve and for $n = 0, 1, ...,$

$$\begin{cases}
x_{12n-2} = c, & y_{12n-2} = g \\
x_{12n-1} = b, & y_{12n-1} = e \\
x_{12n} = a, & y_{12n} = d \\
x_{12n+1} = \dfrac{ag}{g+e}, & y_{12n+1} = \dfrac{dc}{b-c} \\
x_{12n+2} = \dfrac{age}{(g+e)(e+d)}, & y_{12n+2} = \dfrac{dbc}{(b-c)(a-b)} \\
x_{12n+3} = \dfrac{age(b-c)}{b(g+e)(e+d)}, & y_{12n+3} = -\dfrac{dbc(e+g)}{(b-c)(a-b)e} \\
x_{12n+4} = \dfrac{ge(b-c)(a-b)}{b(g+e)(e+d)}, & y_{12n+4} = \dfrac{bc(e+g)(d+e)}{e(b-c)(a-b)} \\
x_{12n+5} = -\dfrac{e^2(b-c)(a-b)}{b(g+e)(e+d)}, & y_{12n+5} = -\dfrac{b^2(e+g)(d+e)}{e(b-c)(a-b)} \\
x_{12n+6} = \dfrac{ed(b-c)(a-b)}{b(g+e)(e+d)}, & y_{12n+6} = \dfrac{ab(e+g)(d+e)}{e(b-c)(a-b)} \\
x_{12n+7} = -\dfrac{edc(a-b)}{b(g+e)(e+d)}, & y_{12n+7} = -\dfrac{abg(d+e)}{e(b-c)(a-b)} \\
x_{12n+8} = \dfrac{edc}{(g+e)(e+d)}, & y_{12n+8} = \dfrac{abg}{(b-c)(a-b)} \\
x_{12n+9} = \dfrac{dc}{(d+e)}, & y_{12n+9} = -\dfrac{ag}{a-b}
\end{cases}$$

where $x_{-2} = c, x_{-1} = b, x_0 = a, y_{-2} = g, y_{-1} = e, y_0 = d$.

Proof: We use an inductive proof for this rational recursive sequences. It is easy to see that for $n = 0$, the result holds. Suppose that $n > 0$ and that the assumption is satisfied for $n - 1$. That is;

$$\begin{cases} x_{12n-14} = c, & y_{12n-14} = g \\ x_{12n-13} = b, & y_{12n-13} = e \\ x_{12n-12} = a, & y_{12n-12} = d \\ x_{12n-11} = \dfrac{ag}{g+e}, & y_{12n-11} = \dfrac{dc}{b-c} \end{cases}$$

$$\begin{cases} x_{12n-10} = \dfrac{age}{(g+e)(e+d)}, & y_{12n-10} = \dfrac{dbc}{(b-c)(a-b)} \\ x_{12n-9} = \dfrac{age(b-c)}{b(g+e)(e+d)}, & y_{12n-9} = -\dfrac{dbc(e+g)}{(b-c)(a-b)e} \\ x_{12n-8} = \dfrac{ge(b-c)(a-b)}{b(g+e)(e+d)}, & y_{12n-8} = \dfrac{bc(e+g)(d+e)}{e(b-c)(a-b)} \\ x_{12n-7} = -\dfrac{e^2(b-c)(a-b)}{b(g+e)(e+d)}, & y_{12n-75} = -\dfrac{b^2(e+g)(d+e)}{e(b-c)(a-b)} \\ x_{12n-6} = \dfrac{ed(b-c)(a-b)}{b(g+e)(e+d)}, & y_{12n-6} = \dfrac{ab(e+g)(d+e)}{e(b-c)(a-b)} \\ x_{12n-5} = -\dfrac{edc(a-b)}{b(g+e)(e+d)}, & y_{12n-5} = -\dfrac{abg(d+e)}{e(b-c)(a-b)} \\ x_{12n-4} = \dfrac{edc}{(g+e)(e+d)}, & y_{12n-4} = \dfrac{abg}{(b-c)(a-b)} \\ x_{12n-3} = \dfrac{dc}{(d+e)}, & y_{12n-3} = -\dfrac{ag}{a-b} \end{cases}$$

Now we find from Eq. (3.1) that

$$
\begin{aligned}
x_{12n-2} &= \frac{x_{12n-3}y_{12n-5}}{y_{12n-4}+y_{12n-5}} \\
&= \frac{-\dfrac{dc}{(d+e)}\dfrac{abg(d+e)}{e(b-c)(a-b)}}{\dfrac{abg}{(b-c)(a-b)}-\dfrac{abg(d+e)}{e(b-c)(a-b)}} \\
&= -\frac{\dfrac{abcdg}{e(b-c)(a-b)}}{\dfrac{abeg}{e(b-c)(a-b)}-\dfrac{abg(d+e)}{e(b-c)(a-b)}} \\
&= -\frac{\dfrac{abcdg}{e(b-c)(a-b)}}{-\dfrac{abgd}{e(b-c)(a-b)}} \\
&= \frac{abcdg}{abgd} \\
&= c
\end{aligned}
$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed.

For confirming the results of this section, we consider numerical example for $x_{-2} = 1, x_{-1} = 2, x_0 = 3, y_{-2} = 3, y_{-1} = 2, y_0 = 1$,(See Figure 2).

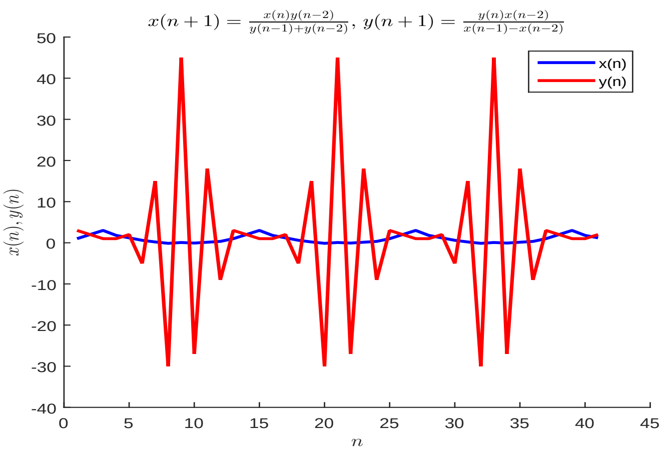

Fig. 2

4 Third Case : On the Difference Equation $x_{n+1} = \dfrac{x_n y_{n-2}}{y_{n-1}+y_{n-2}}, y_{n+1} = \dfrac{y_n x_{n-2}}{-x_{n-1}+x_{n-2}}$

In this section we study the following special case of Eq. (1.1):

$$x_{n+1} = \frac{x_n y_{n-2}}{y_{n-1}+y_{n-2}}, \quad y_{n+1} = \frac{y_n x_{n-2}}{-x_{n-1}+x_{n-2}}, \quad n = 0, 1, ..., \tag{4.1}$$

where the initial conditions $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0$ are arbitrary positive real numbers.

Theorem 5. Let $\{x_n, y_n\}_{n=-2}^{\infty}$ be a solution of Eq. (4.1). Then for $n = 0, 1, ...,$

$$x_{8n-2} = \frac{abcde^2g(a-b)(b-c)}{(af_{2n-3}-bf_{2n-1})(af_{2n-2}-bf_{2n})(bf_{2n-3}-cf_{2n-1})(bf_{2n-2}-cf_{2n})(df_{2n-1}+ef_{2n-2})(df_{2n}+ef_{2n-1})(ef_{2n-1}+gf_{2n-2})(ef_{2n}+gf_{2n-1})},$$

$$x_{8n-1} = \frac{abcde^2g(a-b)(b-c)}{(af_{2n-3}-bf_{2n-1})(af_{2n-2}-bf_{2n})(bf_{2n-2}-cf_{2n})(bf_{2n-1}-cf_{2n+1})(df_{2n-1}+ef_{2n-2})(df_{2n}+ef_{2n-1})(ef_{2n-1}+gf_{2n-2})(ef_{2n}+gf_{2n-1})},$$

$$x_{8n} = \frac{abcde^2g(a-b)(b-c)}{(af_{2n-2}-bf_{2n})(af_{2n-1}-bf_{2n+1})(bf_{2n-2}-cf_{2n})(bf_{2n-1}-cf_{2n+1})(df_{2n-1}+ef_{2n-2})(df_{2n}+ef_{2n-1})(ef_{2n-1}+gf_{2n-2})(ef_{2n}+gf_{2n-1})},$$

$$x_{8n+1} = \frac{abcde^2g(a-b)(b-c)}{(af_{2n-2}-bf_{2n})(af_{2n-1}-bf_{2n+1})(bf_{2n-2}-cf_{2n})(bf_{2n-1}-cf_{2n+1})(df_{2n-1}+ef_{2n-2})(df_{2n}+ef_{2n-1})(ef_{2n}+gf_{2n-1})(ef_{2n+1}+gf_{2n})},$$

$$x_{8n+2} = \frac{abcde^2g(a-b)(b-c)}{(af_{2n-2}-bf_{2n})(af_{2n-1}-bf_{2n+1})(bf_{2n-2}-cf_{2n})(bf_{2n-1}-cf_{2n+1})(df_{2n}+ef_{2n-1})(df_{2n+1}+ef_{2n})(ef_{2n}+gf_{2n-1})(ef_{2n+1}+gf_{2n})},$$

$$x_{8n+3} = \frac{abcde^2g(a-b)(b-c)}{(af_{2n-2}-bf_{2n})(af_{2n-1}-bf_{2n+1})(bf_{2n-1}-cf_{2n+1})(bf_{2n}-cf_{2n+2})(df_{2n}+ef_{2n-1})(df_{2n+1}+ef_{2n})(ef_{2n}+gf_{2n-1})(ef_{2n+1}+gf_{2n})},$$

$$x_{8n+4} = \frac{abcde^2g(a-b)(b-c)}{(af_{2n-1}-bf_{2n+1})(af_{2n}-bf_{2n+2})(bf_{2n-1}-cf_{2n+1})(bf_{2n}-cf_{2n+2})(df_{2n}+ef_{2n-1})(df_{2n+1}+ef_{2n})(ef_{2n}+gf_{2n-1})(ef_{2n+1}+gf_{2n})},$$

$$x_{8n+5} = \frac{abcde^2g(a-b)(b-c)}{(af_{2n-1}-bf_{2n+1})(af_{2n}-bf_{2n+2})(bf_{2n-1}-cf_{2n+1})(bf_{2n}-cf_{2n+2})(df_{2n}+ef_{2n-1})(df_{2n+1}+ef_{2n})(ef_{2n+1}+gf_{2n})(ef_{2n+2}+gf_{2n+1})},$$

and

$$y_{8n-2} = \frac{(af_{2n-2}-bf_{2n})(bf_{2n-2}-cf_{2n})(df_{2n-1}+ef_{2n-2})(ef_{2n-1}+gf_{2n-2})}{e(a-b)(b-c)},$$

$$y_{8n-1} = \frac{(af_{2n-2}-bf_{2n})(bf_{2n-2}-cf_{2n})(df_{2n-1}+ef_{2n})(ef_{2n}+gf_{2n-1})}{e(a-b)(b-c)},$$

$$y_{8n} = \frac{(af_{2n-2}-bf_{2n})(bf_{2n-2}-cf_{2n})(df_{2n}+ef_{2n-1})(ef_{2n}+gf_{2n-1})}{e(a-b)(b-c)},$$

$$y_{8n+1} = \frac{(af_{2n-1}-bf_{2n+1})(bf_{2n-2}-cf_{2n})(df_{2n}+ef_{2n-1})(ef_{2n}+gf_{2n-1})}{e(a-b)(b-c)},$$

$$y_{8n+2} = \frac{(af_{2n-1}-bf_{2n+1})(bf_{2n-1}-cf_{2n+1})(df_{2n}+ef_{2n-1})(ef_{2n}+gf_{2n-1})}{e(a-b)(b-c)},$$

$$y_{8n+3} = \frac{(af_{2n-1}-bf_{2n+1})(bf_{2n-1}-cf_{2n+1})(df_{2n}+ef_{2n-1})(ef_{2n+1}+gf_{2n})}{e(a-b)(b-c)},$$

$$y_{8n+4} = \frac{(af_{2n-1}-bf_{2n+1})(bf_{2n-1}-cf_{2n+1})(df_{2n+1}+ef_{2n})(ef_{2n+1}+gf_{2n})}{e(a-b)(b-c)},$$

$$y_{8n+5} = \frac{(af_{2n-1}-bf_{2n+1})(bf_{2n}-cf_{2n+2})(df_{2n+1}+ef_{2n})(ef_{2n+1}+gf_{2n})}{e(a-b)(b-c)},$$

where $x_{-2} = c, x_{-1} = b, x_0 = a, y_{-2} = g, y_{-1} = e, y_0 = d$ and $\{f_m\}_{m=-2}^{\infty} = \{1, 0, 1, 1, 2, 3, 5, 8, 13, ...\}$.

Proof: We use an inductive proof for this rational recursive sequences. It is easy to see that for

$n = 0$, the result holds. Suppose that $n > 0$ and that the assumption is satisfied for $n-1$. That is;

$$x_{8n-10} = \frac{abcde^2g(a-b)(b-c)}{(af_{2n-5}-bf_{2n-3})(af_{2n-4}-bf_{2n-2})(bf_{2n-5}-cf_{2n-3})(bf_{2n-4}-cf_{2n-2})(df_{2n-3}+ef_{2n-4})(df_{2n-2}+ef_{2n-3})(ef_{2n-3}+gf_{2n-5})(ef_{2n-2}+gf_{2n-3})},$$

$$x_{8n-9} = \frac{abcde^2g(a-b)(b-c)}{(af_{2n-5}-bf_{2n-3})(af_{2n-4}-bf_{2n-2})(bf_{2n-4}-cf_{2n-2})(bf_{2n-3}-cf_{2n-1})(df_{2n-2}+ef_{2n-4})(df_{2n-2}+ef_{2n-3})(ef_{2n-3}+gf_{2n-4})(ef_{2n-2}+gf_{2n-3})},$$

$$x_{8n-8} = \frac{abcde^2g(a-b)(b-c)}{(af_{2n-4}-bf_{2n-2})(af_{2n-3}-bf_{2n-1})(bf_{2n-4}-cf_{2n-2})(bf_{2n-3}-cf_{2n-1})(df_{2n-3}+ef_{2n-4})(df_{2n-2}+ef_{2n-3})(ef_{2n-3}+gf_{2n-4})(ef_{2n-2}+gf_{2n-3})},$$

$$x_{8n-7} = \frac{abcde^2g(a-b)(b-c)}{(af_{2n-4}-bf_{2n-2})(af_{2n-3}-bf_{2n-1})(bf_{2n-4}-cf_{2n-2})(bf_{2n-3}-cf_{2n-1})(df_{2n-3}+ef_{2n-4})(df_{2n-2}+ef_{2n-3})(ef_{2n-2}+gf_{2n-3})(ef_{2n-1}+gf_{2n-2})},$$

$$x_{8n-6} = \frac{abcde^2g(a-b)(b-c)}{(af_{2n-4}-bf_{2n-2})(af_{2n-3}-bf_{2n-1})(bf_{2n-4}-cf_{2n-2})(bf_{2n-3}-cf_{2n-1})(df_{2n-2}+ef_{2n-3})(df_{2n-1}+ef_{2n-2})(ef_{2n-2}+gf_{2n-3})(ef_{2n-1}+gf_{2n-2})},$$

$$x_{8n-5} = \frac{abcde^2g(a-b)(b-c)}{(af_{2n-4}-bf_{2n-2})(af_{2n-3}-bf_{2n-1})(bf_{2n-3}-cf_{2n-1})(bf_{2n-2}-cf_{2n})(df_{2n-2}+ef_{2n-3})(df_{2n-1}+ef_{2n-2})(ef_{2n-2}+gf_{2n-3})(ef_{2n-1}+gf_{2n-2})},$$

$$x_{8n-4} = \frac{abcde^2g(a-b)(b-c)}{(af_{2n-3}-bf_{2n-1})(af_{2n-2}-bf_{2n})(bf_{2n-3}-cf_{2n-1})(bf_{2n-2}-cf_{2n})(df_{2n-2}+ef_{2n-3})(df_{2n-1}+ef_{2n-2})(ef_{2n-2}+gf_{2n-3})(ef_{2n-1}+gf_{2n-2})},$$

$$x_{8n-3} = \frac{abcde^2g(a-b)(b-c)}{(af_{2n-3}-bf_{2n-1})(af_{2n-2}-bf_{2n})(bf_{2n-3}-cf_{2n-1})(bf_{2n-2}-cf_{2n})(df_{2n-2}+ef_{2n-3})(df_{2n-1}+ef_{2n-2})(ef_{2n-1}+gf_{2n-2})(ef_{2n}+gf_{2n-1})},$$

and

$$y_{8n-10} = \frac{(af_{2n-4}-bf_{2n-2})(bf_{2n-4}-cf_{2n-2})(df_{2n-3}+ef_{2n-5})(ef_{2n-3}+gf_{2n-5})}{e(a-b)(b-c)},$$

$$y_{8n-9} = \frac{(af_{2n-4}-bf_{2n-2})(bf_{2n-4}-cf_{2n-2})(df_{2n-3}+ef_{2n-2})(ef_{2n-2}+gf_{2n-3})}{e(a-b)(b-c)},$$

$$y_{8n-8} = \frac{(af_{2n-4}-bf_{2n-2})(bf_{2n-4}-cf_{2n-2})(df_{2n-2}+ef_{2n-3})(ef_{2n-2}+gf_{2n-3})}{e(a-b)(b-c)},$$

$$y_{8n-7} = \frac{(af_{2n-3}-bf_{2n-1})(bf_{2n-4}-cf_{2n-2})(df_{2n-2}+ef_{2n-3})(ef_{2n-2}+gf_{2n-3})}{e(a-b)(b-c)},$$

$$y_{8n-6} = \frac{(af_{2n-3}-bf_{2n-1})(bf_{2n-3}-cf_{2n-1})(df_{2n-2}+ef_{2n-3})(ef_{2n-2}+gf_{2n-3})}{e(a-b)(b-c)},$$

$$y_{8n-5} = \frac{(af_{2n-3}-bf_{2n-1})(bf_{2n-3}-cf_{2n-1})(df_{2n-2}+ef_{2n-3})(ef_{2n-1}+gf_{2n-2})}{e(a-b)(b-c)},$$

$$y_{8n-4} = \frac{(af_{2n-3}-bf_{2n-1})(bf_{2n-3}-cf_{2n-1})(df_{2n-1}+ef_{2n-2})(ef_{2n-1}+gf_{2n-2})}{e(a-b)(b-c)},$$

$$y_{8n-3} = \frac{(af_{2n-3}-bf_{2n-1})(bf_{2n-2}-cf_{2n})(df_{2n-1}+ef_{2n-2})(ef_{2n-1}+gf_{2n-2})}{e(a-b)(b-c)},$$

Now we find from Eq. (4.1) that

$$x_{8n-2} = \frac{x_{8n-3}y_{8n-5}}{y_{8n-4}+y_{8n-5}}$$

$$= \frac{\frac{abcde^2g(a-b)(b-c)}{(af_{2n-3}-bf_{2n-1})(af_{2n-2}-bf_{2n})(bf_{2n-3}-cf_{2n-1})(bf_{2n-2}-cf_{2n})(df_{2n-2}+ef_{2n-3})(df_{2n-1}+ef_{2n-2})(ef_{2n-1}+gf_{2n-2})(ef_{2n}+gf_{2n-1})}\frac{(af_{2n-3}-bf_{2n-1})(bf_{2n-3}-cf_{2n-1})(df_{2n-2}+ef_{2n-3})(ef_{2n-1}+gf_{2n-2})}{e(a-b)(b-c)}}{\frac{(af_{2n-3}-bf_{2n-1})(bf_{2n-3}-cf_{2n-1})(df_{2n-1}+ef_{2n-2})(ef_{2n-1}+gf_{2n-2})}{e(a-b)(b-c)}+\frac{(af_{2n-3}-bf_{2n-1})(bf_{2n-3}-cf_{2n-1})(df_{2n-2}+ef_{2n-3})(ef_{2n-1}+gf_{2n-2})}{e(a-b)(b-c)}}$$

$$= \frac{abcde^2g(a-b)(b-c)}{(af_{2n-3}-bf_{2n-1})(af_{2n-2}-bf_{2n})(bf_{2n-3}-cf_{2n-1})(bf_{2n-2}-cf_{2n})(df_{2n-2}+ef_{2n-3})(df_{2n-1}+ef_{2n-2})(ef_{2n-1}+gf_{2n-2})(ef_{2n}+gf_{2n-1})} \frac{af_{2n-3}-bf_{2n-1})(bf_{2n-3}-cf_{2n-1})(df_{2n-2}+ef_{2n-3})(ef_{2n-1}+gf_{2n-2})}{(af_{2n-3}-bf_{2n-1})(bf_{2n-3}-cf_{2n-1})(df_{2n-1}+ef_{2n-2})(ef_{2n-1}+gf_{2n-2})+(af_{2n-3}-bf_{2n-1})(bf_{2n-3}-cf_{2n-1})(df_{2n-2}+ef_{2n-3})(ef_{2n-1}+gf_{2n-2})}$$

$$= \frac{abcde^2g(a-b)(b-c)}{(af_{2n-3}-bf_{2n-1})(af_{2n-2}-bf_{2n})(bf_{2n-3}-cf_{2n-1})(bf_{2n-2}-cf_{2n})(df_{2n-2}+ef_{2n-3})(df_{2n-1}+ef_{2n-2})(ef_{2n-1}+gf_{2n-2})(ef_{2n}+gf_{2n-1})} \frac{(af_{2n-3}-bf_{2n-1})(bf_{2n-3}-cf_{2n-1})(df_{2n-2}+ef_{2n-3})(ef_{2n-1}+gf_{2n-2})}{(af_{2n-3}-bf_{2n-1})(bf_{2n-3}-cf_{2n-1})(df_{2n}+ef_{2n-1})(ef_{2n-1}+gf_{2n-2})}$$

$$= \frac{abcde^2g(a-b)(b-c)}{(af_{2n-3}-bf_{2n-1})(af_{2n-2}-bf_{2n})(bf_{2n-3}-cf_{2n-1})(bf_{2n-2}-cf_{2n})(df_{2n-2}+ef_{2n-3})(df_{2n-1}+ef_{2n-2})(ef_{2n-1}+gf_{2n-2})(ef_{2n}+gf_{2n-1})} \frac{(df_{2n-2}+ef_{2n-3})}{(df_{2n}+ef_{2n-1})}$$

$$= \frac{abcde^2g(a-b)(b-c)}{(af_{2n-3}-bf_{2n-1})(af_{2n-2}-bf_{2n})(bf_{2n-3}-cf_{2n-1})(bf_{2n-2}-cf_{2n})(df_{2n}+ef_{2n-1})(df_{2n-1}+ef_{2n-2})(ef_{2n-1}+gf_{2n-2})(ef_{2n}+gf_{2n-1})}$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed.

For confirming the results of this section, we consider numerical example for $x_{-2} = -1, x_{-1} = 2, x_0 = -9, y_{-2} = 1, y_{-1} = 2, y_0 = 1$,(See Figure 1).

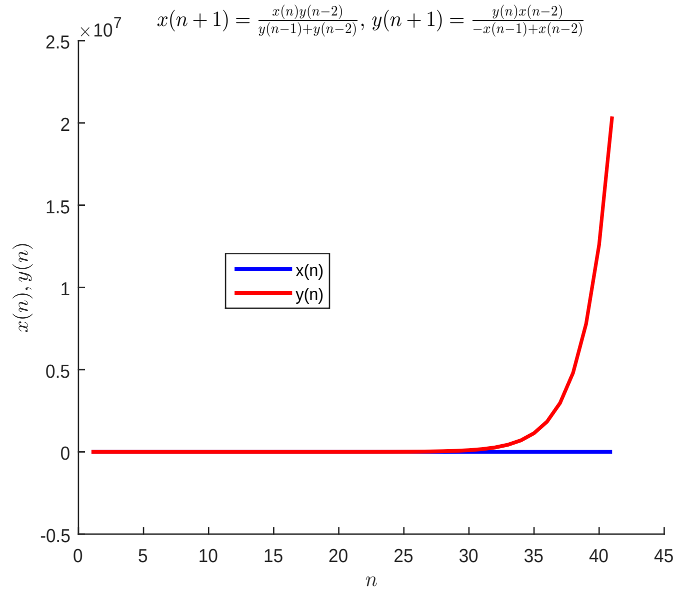

Fig. 3

5 Fourth Case : On the Difference Equation $x_{n+1} = \frac{x_n y_{n-2}}{y_{n-1} + y_{n-2}}, \quad y_{n+1} = -\frac{y_n x_{n-2}}{x_{n-1} + x_{n-2}}$

In this section we study the following special case of Eq. (1.1):

$$x_{n+1} = \frac{x_n y_{n-2}}{y_{n-1} + y_{n-2}}, \quad y_{n+1} = -\frac{y_n x_{n-2}}{x_{n-1} + x_{n-2}}, \quad n = 0, 1, ..., \tag{5.1}$$

where the initial conditions $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0$ are arbitrary positive real numbers.

Theorem 5. Let $\{x_n, y_n\}_{n=-2}^{\infty}$ be a solution of Eq. (5.1). Then for $n = 0, 1, ...,$

$$x_{8n-2} = \frac{(af_{2n-1} + bf_{2n-2})(bf_{2n-1} + cf_{2n-2})(df_{2n-2} + ef_{2n})(ef_{2n-2} + gf_{2n})}{b(d+e)(e+g)},$$

$$x_{8n-1} = \frac{(af_{2n-1} + bf_{2n-2})(bf_{2n} + cf_{2n-1})(df_{2n-2} + ef_{2n})(ef_{2n-2} + gf_{2n})}{b(d+e)(e+g)},$$

$$x_{8n} = \frac{(af_{2n} + bf_{2n-1})(bf_{2n} + cf_{2n-1})(df_{2n-2} + ef_{2n})(ef_{2n-2} + gf_{2n})}{b(d+e)(e+g)},$$

$$x_{8n+1} = \frac{(af_{2n} + bf_{2n-1})(bf_{2n} + cf_{2n-1})(df_{2n-2} + ef_{2n})(ef_{2n-1} + gf_{2n+1})}{b(d+e)(e+g)},$$

$$x_{8n+2} = \frac{(af_{2n} + bf_{2n-1})(bf_{2n} + cf_{2n-1})(df_{2n-1} + ef_{2n+1})(ef_{2n-1} + gf_{2n+1})}{b(d+e)(e+g)},$$

$$x_{8n+3} = \frac{(af_{2n} + bf_{2n-1})(bf_{2n+1} + cf_{2n})(df_{2n-1} + ef_{2n+1})(ef_{2n-1} + gf_{2n+1})}{b(d+e)(e+g)},$$

$$x_{8n+4} = \frac{(af_{2n+1} + bf_{2n})(bf_{2n+1} + cf_{2n})(df_{2n-1} + ef_{2n+1})(ef_{2n-1} + gf_{2n+1})}{b(d+e)(e+g)},$$

$$x_{8n+5} = \frac{(af_{2n+1} + bf_{2n})(bf_{2n+1} + cf_{2n})(df_{2n-1} + ef_{2n+1})(ef_{2n} + gf_{2n+2})}{b(d+e)(e+g)},$$

and

$$y_{8n-2} = \frac{ab^2cdeg(e+g)(d+e)}{(af_{2n-1} + bf_{2n-2})(af_{2n} + bf_{2n-1})(bf_{2n-1} + cf_{2n-2})(bf_{2n} + cf_{2n-1})(df_{2n-3} + ef_{2n-1})(df_{2n-2} + ef_{2n})(ef_{2n-3} + gf_{2n-1})(ef_{2n-2} + gf_{2n})},$$

$$y_{8n-1} = -\frac{ab^2cdeg(e+g)(d+e)}{(af_{2n-1} + bf_{2n-2})(af_{2n} + bf_{2n-1})(bf_{2n-1} + cf_{2n-2})(bf_{2n} + cf_{2n-1})(df_{2n-3} + ef_{2n-1})(df_{2n-2} + ef_{2n})(ef_{2n-2} + gf_{2n})(ef_{2n-1} + gf_{2n+1})},$$

$$y_{8n} = \frac{ab^2cdeg(e+g)(d+e)}{(af_{2n-1} + bf_{2n-2})(af_{2n} + bf_{2n-1})(bf_{2n-1} + cf_{2n-2})(bf_{2n} + cf_{2n-1})(df_{2n-2} + ef_{2n})(df_{2n-1} + ef_{2n+1})(ef_{2n-2} + gf_{2n})(ef_{2n-1} + gf_{2n+1})},$$

$$y_{8n+1} = -\frac{ab^2cdeg(e+g)(d+e)}{(af_{2n-1} + bf_{2n-2})(af_{2n} + bf_{2n-1})(bf_{2n} + cf_{2n-1})(bf_{2n+1} + cf_{2n})(df_{2n-2} + ef_{2n})(df_{2n-1} + ef_{2n+1})(ef_{2n-2} + gf_{2n})(ef_{2n-1} + gf_{2n+1})},$$

$$y_{8n+2} = \frac{ab^2cdeg(e+g)(d+e)}{(af_{2n} + bf_{2n-1})(af_{2n+1} + bf_{2n})(bf_{2n} + cf_{2n-1})(bf_{2n+1} + cf_{2n})(df_{2n-2} + ef_{2n})(df_{2n-1} + ef_{2n+1})(ef_{2n-2} + gf_{2n})(ef_{2n-1} + gf_{2n+1})},$$

$$y_{8n+3} = -\frac{ab^2cdeg(e+g)(d+e)}{(af_{2n} + bf_{2n-1})(af_{2n+1} + bf_{2n})(bf_{2n} + cf_{2n-1})(bf_{2n+1} + cf_{2n})(df_{2n-2} + ef_{2n})(df_{2n-1} + ef_{2n+1})(ef_{2n-1} + gf_{2n+1})(ef_{2n} + gf_{2n+2})},$$

$$y_{8n+4} = \frac{ab^2cdeg(e+g)(d+e)}{(af_{2n} + bf_{2n-1})(af_{2n+1} + bf_{2n})(bf_{2n} + cf_{2n-1})(bf_{2n+1} + cf_{2n})(df_{2n-1} + ef_{2n+1})(df_{2n} + ef_{2n+2})(ef_{2n-1} + gf_{2n+1})(ef_{2n} + gf_{2n+2})},$$

$$y_{8n+5} = -\frac{ab^2cdeg(e+g)(d+e)}{(af_{2n} + bf_{2n-1})(af_{2n+1} + bf_{2n})(bf_{2n+1} + cf_{2n})(bf_{2n+2} + cf_{2n+1})(df_{2n-1} + ef_{2n+1})(df_{2n} + ef_{2n+2})(ef_{2n-1} + gf_{2n+1})(ef_{2n} + gf_{2n+2})},$$

where $x_{-2} = c, x_{-1} = b, x_0 = a, y_{-2} = g, y_{-1} = e, y_0 = d$ and $\{f_m\}_{m=-2}^{\infty} = \{1, 0, 1, 1, 2, 3, 5, 8, 13, ...\}$.

Proof: We use an inductive proof for this rational recursive sequences. It is easy to see that for $n = 0$, the result holds. Suppose that $n > 0$ and that the assumption is satisfied for $n - 1$. That is;

$$x_{8n-10} = \frac{(af_{2n-3} + bf_{2n-4})(bf_{2n-3} + cf_{2n-4})(df_{2n-4} + ef_{2n-2})(ef_{2n-4} + gf_{2n-2})}{b(d+e)(e+g)},$$

$$x_{8n-9} = \frac{(af_{2n-3} + bf_{2n-4})(bf_{2n-2} + cf_{2n-3})(df_{2n-4} + ef_{2n-2})(ef_{2n-4} + gf_{2n-2})}{b(d+e)(e+g)},$$

$$x_{8n-8} = \frac{(af_{2n-2} + bf_{2n-3})(bf_{2n-2} + cf_{2n-3})(df_{2n-4} + ef_{2n-2})(ef_{2n-4} + gf_{2n-2})}{b(d+e)(e+g)},$$

$$x_{8n-7} = \frac{(af_{2n-2} + bf_{2n-3})(bf_{2n-2} + cf_{2n-3})(df_{2n-4} + ef_{2n-2})(ef_{2n-3} + gf_{2n-1})}{b(d+e)(e+g)},$$

$$x_{8n-6} = \frac{(af_{2n-2} + bf_{2n-3})(bf_{2n-2} + cf_{2n-3})(df_{2n-3} + ef_{2n-1})(ef_{2n-3} + gf_{2n-1})}{b(d+e)(e+g)},$$

$$x_{8n-5} = \frac{(af_{2n-2} + bf_{2n-3})(bf_{2n-1} + cf_{2n-2})(df_{2n-3} + ef_{2n-1})(ef_{2n-3} + gf_{2n-1})}{b(d+e)(e+g)},$$

$$x_{8n-4} = \frac{(af_{2n-1} + bf_{2n-2})(bf_{2n-1} + cf_{2n-2})(df_{2n-3} + ef_{2n-1})(ef_{2n-3} + gf_{2n-1})}{b(d+e)(e+g)},$$

$$x_{8n-3} = \frac{(af_{2n-1} + bf_{2n-2})(bf_{2n-1} + cf_{2n-2})(df_{2n-3} + ef_{2n-1})(ef_{2n-2} + gf_{2n})}{b(d+e)(e+g)},$$

and

$$y_{8n-10} = \frac{ab^2cdeg(e+g)(d+e)}{(af_{2n-3} + bf_{2n-4})(af_{2n-2} + bf_{2n-3})(bf_{2n-3} + cf_{2n-4})(bf_{2n-2} + cf_{2n-3})(df_{2n-5} + ef_{2n-3})(df_{2n-4} + ef_{2n-2})(ef_{2n-5} + gf_{2n-3})(ef_{2n-4} + gf_{2n-2})},$$

$$y_{8n-9} = -\frac{ab^2cdeg(e+g)(d+e)}{(af_{2n-3} + bf_{2n-4})(af_{2n-2} + bf_{2n-3})(bf_{2n-3} + cf_{2n-4})(bf_{2n-2} + cf_{2n-3})(df_{2n-5} + ef_{2n-3})(df_{2n-4} + ef_{2n-2})(ef_{2n-4} + gf_{2n-2})(ef_{2n-3} + gf_{2n-1})},$$

$$y_{8n-8} = \frac{ab^2cdeg(e+g)(d+e)}{(af_{2n-3} + bf_{2n-4})(af_{2n-2} + bf_{2n-3})(bf_{2n-3} + cf_{2n-4})(bf_{2n-2} + cf_{2n-3})(df_{2n-4} + ef_{2n-2})(df_{2n-3} + ef_{2n-1})(ef_{2n-4} + gf_{2n-2})(ef_{2n-3} + gf_{2n-1})},$$

$$y_{8n-7} = -\frac{ab^2cdeg(e+g)(d+e)}{(af_{2n-3} + bf_{2n-4})(af_{2n-2} + bf_{2n-3})(bf_{2n-2} + cf_{2n-3})(bf_{2n-1} + cf_{2n-2})(df_{2n-4} + ef_{2n-2})(df_{2n-3} + ef_{2n-1})(ef_{2n-4} + gf_{2n-2})(ef_{2n-3} + gf_{2n-1})},$$

$$y_{8n-6} = \frac{ab^2cdeg(e+g)(d+e)}{(af_{2n-2} + bf_{2n-3})(af_{2n-1} + bf_{2n-2})(bf_{2n-2} + cf_{2n-3})(bf_{2n-1} + cf_{2n-2})(df_{2n-4} + ef_{2n-2})(df_{2n-3} + ef_{2n-1})(ef_{2n-4} + gf_{2n-2})(ef_{2n-3} + gf_{2n-1})},$$

$$y_{8n-5} = -\frac{ab^2cdeg(e+g)(d+e)}{(af_{2n-2} + bf_{2n-3})(af_{2n-1} + bf_{2n-2})(bf_{2n-2} + cf_{2n-3})(bf_{2n-1} + cf_{2n-2})(df_{2n-4} + ef_{2n-2})(df_{2n-3} + ef_{2n-1})(ef_{2n-3} + gf_{2n-1})(ef_{2n-2} + gf_{2n})},$$

$$y_{8n-4} = \frac{ab^2cdeg(e+g)(d+e)}{(af_{2n-2} + bf_{2n-3})(af_{2n-1} + bf_{2n-2})(bf_{2n-2} + cf_{2n-3})(bf_{2n-1} + cf_{2n-2})(df_{2n-3} + ef_{2n-1})(df_{2n-2} + ef_{2n})(ef_{2n-3} + gf_{2n-1})(ef_{2n-2} + gf_{2n})},$$

$$y_{8n-3} = -\frac{ab^2cdeg(e+g)(d+e)}{(af_{2n-2} + bf_{2n-3})(af_{2n-1} + bf_{2n-2})(bf_{2n-1} + cf_{2n-2})(bf_{2n} + cf_{2n-1})(df_{2n-3} + ef_{2n-1})(df_{2n-2} + ef_{2n})(ef_{2n-3} + gf_{2n-1})(ef_{2n-2} + gf_{2n})},$$

Now we find from Eq. (5.1) that

$$x_{8n-2} = -\frac{x_{8n-3}y_{8n-5}}{y_{8n-4}+y_{8n-5}}$$

$$= \frac{-\frac{(af_{2n-1}+bf_{2n-2})(bf_{2n-1}+cf_{2n-2})(df_{2n-3}+ef_{2n-1})(ef_{2n-2}+gf_{2n})}{b(d+e)(e+g)}\frac{ab^2cdeg(e+g)(d+e)}{(af_{2n-2}+bf_{2n-3})(af_{2n-1}+bf_{2n-2})(bf_{2n-2}+cf_{2n-3})(bf_{2n-1}+cf_{2n-2})(df_{2n-4}+ef_{2n-2})(df_{2n-3}+ef_{2n-1})(ef_{2n-3}+gf_{2n-1})(ef_{2n-2}+gf_{2n})}}{\frac{ab^2cdeg(e+g)(d+e)}{(af_{2n-2}+bf_{2n-3})(af_{2n-1}+bf_{2n-2})(bf_{2n-2}+cf_{2n-3})(bf_{2n-1}+cf_{2n-2})(df_{2n-3}+ef_{2n-1})(df_{2n-2}+ef_{2n})(ef_{2n-3}+gf_{2n-1})(ef_{2n-2}+gf_{2n})} - \frac{ab^2cdeg(e+g)(d+e)}{(af_{2n-2}+bf_{2n-3})(af_{2n-1}+bf_{2n-2})(bf_{2n-2}+cf_{2n-3})(bf_{2n-1}+cf_{2n-2})(df_{2n-4}+ef_{2n-2})(df_{2n-3}+ef_{2n-1})(ef_{2n-3}+gf_{2n-1})(ef_{2n-2}+gf_{2n})}}$$

$$= \frac{-\frac{abcdeg}{(af_{2n-2}+bf_{2n-3})(bf_{2n-2}+cf_{2n-3})(df_{2n-4}+ef_{2n-2})(ef_{2n-3}+gf_{2n-1})}}{\frac{ab^2cdeg(e+g)(d+e)}{(af_{2n-2}+bf_{2n-3})(af_{2n-1}+bf_{2n-2})(bf_{2n-2}+cf_{2n-3})(bf_{2n-1}+cf_{2n-2})(df_{2n-3}+ef_{2n-1})(ef_{2n-3}+gf_{2n-1})(ef_{2n-2}+gf_{2n})}\left(\frac{1}{(df_{2n-2}+ef_{2n})} - \frac{1}{(df_{2n-4}+ef_{2n-2})}\right)}$$

$$= \frac{-\frac{1}{(df_{2n-4}+ef_{2n-2})}}{\frac{b(e+g)(d+e)}{(af_{2n-1}+bf_{2n-2})(bf_{2n-1}+cf_{2n-2})(df_{2n-3}+ef_{2n-1})(ef_{2n-2}+gf_{2n})}\left(\frac{1}{(df_{2n-2}+ef_{2n})} - \frac{1}{(df_{2n-4}+ef_{2n-2})}\right)}$$

$$= \frac{-\frac{1}{(df_{2n-4}+ef_{2n-2})}}{\frac{b(e+g)(d+e)}{(af_{2n-1}+bf_{2n-2})(bf_{2n-1}+cf_{2n-2})(df_{2n-3}+ef_{2n-1})(ef_{2n-2}+gf_{2n})}\frac{df_{2n-4}+ef_{2n-2}-df_{2n-2}-ef_{2n}}{(df_{2n-2}+ef_{2n})(df_{2n-4}+ef_{2n-2})}}$$

$$= \frac{-1}{\frac{b(e+g)(d+e)(df_{2n-4}+ef_{2n-2}-df_{2n-2}-ef_{2n})}{(af_{2n-1}+bf_{2n-2})(bf_{2n-1}+cf_{2n-2})(df_{2n-3}+ef_{2n-1})(ef_{2n-2}+gf_{2n})(df_{2n-2}+ef_{2n})}}$$

$$= -\frac{(af_{2n-1}+bf_{2n-2})(bf_{2n-1}+cf_{2n-2})(df_{2n-3}+ef_{2n-1})(ef_{2n-2}+gf_{2n})(df_{2n-2}+ef_{2n})}{b(e+g)(d+e)(df_{2n-4}+ef_{2n-2}-df_{2n-2}-ef_{2n})}$$

$$= -\frac{(af_{2n-1}+bf_{2n-2})(bf_{2n-1}+cf_{2n-2})(df_{2n-3}+ef_{2n-1})(ef_{2n-2}+gf_{2n})(df_{2n-2}+ef_{2n})}{b(e+g)(d+e)(-df_{2n-3}-ef_{2n-1})}$$

$$= \frac{(af_{2n-1}+bf_{2n-2})(bf_{2n-1}+cf_{2n-2})(ef_{2n-2}+gf_{2n})(df_{2n-2}+ef_{2n})}{b(e+g)(d+e)}$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed.

For confirming the results of this section, we consider numerical example for $x_{-2} = 1, x_{-1} = -2, x_0 = 6, y_{-2} = 10, y_{-1} = 2, y_0 = 1$,(See Figure 4).

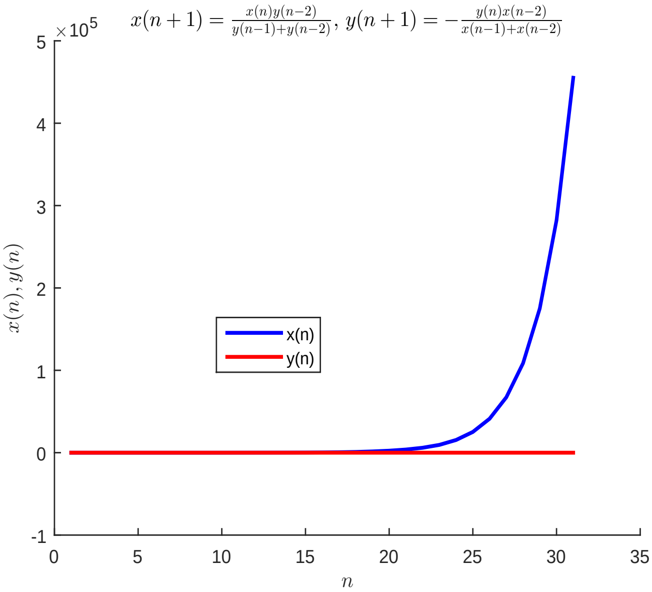

Fig. 4

6 Conclusion

This paper discussed local stability, the solutions of some special cases of Eq. (1.1) and gave numerical examples of each case.

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Competing Interests

Authors have declared that no competing interests exist.

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