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Second Derivative Two-step Block Hybrid Enright's Linear Multistep Methods for Solving Initial Value Problems of General Second Order Stiff Ordinary Differential Equations

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Authors' contributions

This work was carried out in collaboration between all authors. Author SJ designed the method. Author YTK analyzed the basic properties of the method and author AAB implemented the method on some stiff differential equations. All authors read and approved the final manuscript.

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Abstract

In this research, the formation of second derivative two-step block hybrid Enright's linear multistep methods for solving initial value problems is studied. In forming the method, we follow Enright's 1974 approach, by introducing the off-mesh points at both interpolation and collocations; we developed the continuous schemes for new Enright's method. The analysis of new Enright method was studied and it was found to be consistent, convergent and zero-stable. We further computed the order, error constants and plotted the region of absolute stability within which the method is A-stable. The methods exhibited better accuracy level when provided with numerical examples than the existing method with which we compared our results.

Keywords: Enright; block hybrid LMMs; IVPs; second order ODEs; interpolation and collocations.

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1 Introduction

Recently, the integration of Ordinary Differential Equations (ODEs) is investigated using different types of block methods. This present work discusses the formation of implicit Linear Multistep Method (LMM) for numerical integration of general second order ODEs which arise frequently in the area of science and engineering especially mechanical system, control theory and celestial mechanics [1].

$$y''=f(x,y',y),\quad y(a)=y_0,\quad y'(a)=\eta_0,\quad x\in[a,b] \tag{1.1}$$

Many researchers have developed numerical methods for solving equation of the form (1.1) using Enright Approach, [2], [3], [4] and [5].

These techniques have been introduced in many literature such as [6], [2], [7], and others. Most practical problems that arise in various fields, like engineering or science are considered as mathematical models before being solved. These models often lead to differential equations, which can be defined as an equation containing a derivative. Numerous problems such as orbital dynamics, chemical kinetics, circuit and control theory and Newton's second law applications involve second-order ODEs as [6]. In many diverse fields such as operation research, engineering, behavioural sciences, industrial mathematics, artificial intelligence, management and sociology Ordinary differential equations (ODEs) are commonly used for mathematical modelling [1]. Mathematical modelling is the art of translating problem from an application area into tractable mathematical formulations whose numerical and theoretical analysis provides answers, insight and guidance useful for the originating application [8]. This type of problem can be formulated as first-order or higher order ODEs.

This research is organised as follows: in section 2, the derivation of the method was discussed, where we considered two off-step points through the approach of interpolation and collocation. In Section three, the analysis of the methods was discussed. In section four, a few numerical problems were solved and the performance of the developed method was compared with the existing methods [9] and [10]. Finally, in the fifth section, the conclusion was drawn.

2 Derivation of the Method

Very often, reactions in physical systems transform into system of ODE. Some classes of these systems are called Stiff system. The modification of second derivative linear multistep ordinary differential equation for solving stiffly differential equation was studied by Sabo et al, 2018. The numerical methods to obtain solutions to class of problems are one-step method and Multistep method (MM), (Adeniran, Odejide & Ogundare, 2015). Using interpolation and collocation technique, the second derivative multistep methods are derived [3], (Enright &Hull, 1975) and [4]. Consider the initial value problem of the form

$$y''=f(x,y',y),\quad y(a)=y_0,\quad y'(a)=\eta_0,\quad x\in[a,b] \tag{2.1}$$

The general second derivative formula for solving equation (2.1) using $k-step$ second derivative linear multistep method is of the form

$$f(x,y,y')=\sum_{j=0}^{k}\alpha_j y_{n+j}+h\sum_{j=0}^{k}\beta_j f_{n+j}+h^2\sum_{j=0}^{k}\delta_j g_{n+j} \tag{2.2}$$

Where

$$y_{n+j}=y(x_{n+jh})$$

$$f_{n+j} = f\left(x_{n+jh}, \quad y\left(x_{n+jh}\right)\right)$$
$$g_{n+j} = df\left(x_n, \quad y(x)\right)$$
$$g_{n+j} = \left.\frac{df\left(x_n, \quad y(x)\right)}{dx}\right\rangle_{y=y_{n+j}}^{x=y_{n+j}}$$

x_n is a discrete point at x *and* $\alpha_j, \quad \beta_j, \quad \gamma_j$ are coefficients to be determined. To obtained the method of the form (2.2), $y(x)$ is approximated by a basis polynomial of the form

$$y(x) = \sum_{j=0}^{m} \alpha_j \left(\frac{x - x_n}{h}\right)^j \tag{2.3}$$

equation (2.3) will be used for the derivation of the main and complementary methods for the class of continuous second derivatives multistep method of [3] which is a special case of (2.3). Interpolating $y(x)$ at point x_n , collocating $y^{'}(x)$ at point $x_n, x_{n+1}, x_{n+\frac{4}{3}}, \ x_{n+\frac{5}{3}} \ and x_{n+2}$ x_n and collocating $y^{"}(x)$ at points $x_n, x_{n+1}, x_{n+\frac{4}{3}}, \ x_{n+\frac{5}{3}} \ and x_{n+2}$.

$$\left.\begin{array}{l} y^{'}(x) = f_{n+j} \\ y''(x) = f_{n+j} \end{array}\right\rangle j = 0, 1, 2, \ldots, k$$

The system of equations generated are solved to obtained the coefficients of $\alpha_j, \quad j = 0, 1, 2, ..., k+2$ which are used to generate the continuous multistep method of Enright of the form

$$y(x) = y_{n+k-1} + h\sum_{j=0}^{k} \beta_j f_{n+j} + h^2 \delta_k g_{n+k} \tag{2.4}$$

evaluating (2.4) $x = x_{n+k}$ yields the second derivative multistep method of Enright, evaluating at $x = x_{n+j}, \ j = 0, 1, 2, \ldots, k-2$ gives $(k-1)$ methods, which will be called complementary methods to complete the k block for the system. The Enright's method so obtained is of the form

$$y_{n+k} = y_{n+k-1} + h\sum_{j=0}^{k} \beta_j f_{n+j} + h^2 \delta_k g_{n+k} \tag{2.5}$$

To derive the continuous second derivative multistep method of Enright, Let the basis function $y(x)$ be

$$y''(x) = \sum_{j=0}^{k} \left(\frac{x - x_n}{h}\right)^j \tag{2.6}$$

We interpolate (2.6) at point $x = x_n$ collocate $y'(x)$ at points $x_n, x_{n+1}, x_{n+\frac{4}{3}}, x_{n+\frac{5}{3}}$ $and\, x_{n+2}$, and $y''(x)$ at points $x_n, x_{n+1}, x_{n+\frac{4}{3}}, x_{n+\frac{5}{3}}$ $and\, x_{n+2}$, we obtain a system of equation represented in matrix form

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{h} & \frac{2}{h} & \frac{3}{h} & \frac{4}{h} & \frac{5}{h} & \frac{6}{h} & \frac{7}{h} & \frac{8}{h} & \frac{9}{h} & \frac{10}{h} \\
0 & \frac{1}{h} & \frac{8}{3h} & \frac{16}{3h} & \frac{256}{27h} & \frac{128}{81h} & \frac{2048}{81h} & \frac{286872}{729h} & \frac{13072}{2187h} & \frac{65536}{729h} & \frac{2621440}{19683h} \\
0 & \frac{1}{h} & \frac{10}{3h} & \frac{25}{3h} & \frac{500}{27h} & \frac{3125}{81h} & \frac{6250}{81h} & \frac{109375}{729h} & \frac{62500}{2187h} & \frac{390625}{729h} & \frac{19531250}{19683h} \\
0 & \frac{1}{h} & \frac{4}{h} & \frac{12}{h} & \frac{32}{h} & \frac{80}{h} & \frac{192}{h} & \frac{448}{h} & \frac{1024}{h} & \frac{2304}{h} & \frac{5120}{h} \\
0 & 0 & \frac{2}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{2}{h^2} & \frac{6}{h^2} & \frac{12}{h^2} & \frac{20}{h^2} & \frac{30}{h^2} & \frac{42}{h^2} & \frac{56}{h^2} & \frac{72}{h^2} & \frac{90}{h^2} \\
0 & 0 & \frac{2}{h^2} & \frac{8}{h^2} & \frac{64}{3h^2} & \frac{1280}{27h^2} & \frac{2560}{27h^2} & \frac{14336}{81h^2} & \frac{229376}{729h^2} & \frac{131072}{243h^2} & \frac{655360}{729h^2} \\
0 & 0 & \frac{2}{h^2} & \frac{10}{h^2} & \frac{100}{3h^2} & \frac{2500}{27h^2} & \frac{6250}{27h^2} & \frac{43750}{81h^2} & \frac{875000}{729h^2} & \frac{625000}{243h^2} & \frac{3906250}{729h^2} \\
0 & 0 & \frac{2}{h^2} & \frac{12}{h^2} & \frac{48}{h^2} & \frac{160}{h^2} & \frac{480}{h^2} & \frac{1344}{h^2} & \frac{3584}{h^2} & \frac{9216}{h^2} & \frac{23040}{h^2}
\end{pmatrix}
\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \end{pmatrix}
=
\begin{pmatrix} {}^j y_n \\ {}^j y_{n+1} \\ {}^j y_{n+\frac{4}{3}} \\ {}^j y_{n+\frac{5}{3}} \\ {}^j y_{n+2} \\ {}^j f_n \\ {}^j f_{n+1} \\ {}^j f_{n+\frac{4}{3}1} \\ {}^j f_{n+\frac{5}{2}} \\ {}^j f_{n+2} \end{pmatrix} \qquad (2.7)$$

$j = 1, \cdots, m.$

Applying the Gaussian elimination method on Equation (2.7) gives the coefficient $a_i's, \quad for \quad i = 0\,(1)\,10$.

These values are then substituted into Equation (2.5) to give the implicit continuous hybrid method of the form:

$$^{j}y(x) = \sum_{i=0} {}^{j}\alpha_i(x)^j f_{n+i} + h \sum_{i=1,\frac{4}{3},\frac{5}{3},2}^{2} {}^{j}\beta_i(x)^j f_{n+i} + h^2 \sum_{i=1,\frac{4}{3},\frac{5}{3},2}^{2} {}^{j}\beta_i(x)^j f_{n+i}, \quad j = 1, \cdots, m. \qquad (2.8)$$

differentiating Equation (2.8) once yields:

$$^{j}y''(x) = \sum_{i=0} \frac{d}{dx} {}^{j}\alpha_i(x)^j f_{n+i} + h \sum_{i=1,\frac{4}{3},\frac{5}{3},2}^{2} \frac{d}{dx} {}^{j}\beta_i(x)^j f_{n+i} + h^2 \sum_{i=1,\frac{4}{3},\frac{5}{3},2}^{2} \frac{d}{dx} {}^{j}\beta_i(x)^j f_{n+i}, \quad j = 1, \cdots, m \qquad (2.9)$$

where the continuous schemes are

$$^{j}\alpha_0 = 0$$

$$\beta_0 = th + \frac{1}{6720000} t^3 h \begin{pmatrix} -41255200 + 101733660t - 127750056t^2 + 98067270t^3 - 47936880t^4 + 14625765t^5 \\ -2547720t^6 + 193914t^7 \end{pmatrix}$$

$$\beta_1 = \frac{1}{33600} t^3 h \begin{pmatrix} -35840000 + 129696000\,t - 211491840\,t^2 + 197013600\,t^3 - 111931200\,t^4 + 3852650\,t^5 \\ -7408800\,t^6 + 11160261\,t^7 \end{pmatrix}$$

$$\beta_{\frac{4}{3}} = \frac{243}{17920} t^3 h \begin{pmatrix} -28000 + 119700\,t - 226968\,t^2 + 242130\,t^3 - 155280\,t^4 + 59535\,t^5 \\ -12600\,t^6 + 1134\,t^7 \end{pmatrix}$$

$$\beta_{\frac{5}{3}} = \frac{243}{140000} t^3 h \begin{pmatrix} -716800 + 2701440\,t - 4591104\,t^2 + 4453680\,t^3 - 2629920\,t^4 + 938385\,t^5 \\ -186480\,t^6 + 15876\,t^7 \end{pmatrix}$$

$$\beta_2 = \frac{1}{1680} t^3 h \begin{pmatrix} -350000 + 1359750\,t - 2389485\,t^2 + 2404290\,t^3 - 1477170\,t^4 + 549990\,t^5 \\ -114345\,t^6 + 10206\,t^7 \end{pmatrix}$$

$$\gamma_0 = \frac{1}{672000} t^2 h^2 \begin{pmatrix} 336000 - 1276800\,t + 2364180\,t^2 - 2641968\,t^3 + 1903510\,t^4 - 894240\,t^5 \\ +265545\,t^6 - 45360\,t^7 + 3402\,t^8 \end{pmatrix}$$

$$\gamma_1 = \frac{1}{3360} t^3 h^2 \begin{pmatrix} -448000 + 1579200\,t - 2519328\,t^2 + 2303840\,t^3 - 1288440\,t^4 + 437535\,t^5 \\ -83160\,t^6 + 6804\,t^7 \end{pmatrix}$$

$$\gamma_{\frac{4}{3}} = \frac{81}{8960} t^3 h^2 \begin{pmatrix} -56000 + 207900\,t - 348096\,t^2 + 332850\,t^3 - 193920\,t^4 + 68355\,t^5 \\ -13440\,t^6 + 1134\,t^7 \end{pmatrix}$$

$$\gamma_{\frac{5}{3}} = \frac{81}{4000} t^3 h^2 \begin{pmatrix} -12800 + 48960\,t - 84576\,t^2 + 83520\,t^3 - 50280\,t^4 + 18315\,t^5 \\ -3720\,t^6 + 324\,t^7 \end{pmatrix}$$

$$\gamma_2 = \frac{1}{6720} t^3 h^2 \begin{pmatrix} -112000 + 436800\,t - 770952\,t^2 + 779590\,t^3 - 481680\,t^4 + 180495\,t^5 \\ -37800\,t^6 + 3402\,t^7 \end{pmatrix}$$

3 Convergence Analysis of new Enright's Method

3.1 Order and error constants of the methods

Using the Taylor series, the order of the new method in Equation (6) is obtained [11], it is found that the developed method has uniformly order ten, with an error constants vector of:

$$C_{11} = \left[2.8616 \times 10^{-8},\, 2.8625 \times 10^{-8},\, 2.8631 \times 10^{-8},\, 2.8655 \times 10^{-8}\right]^T$$

3.2 Consistency

Definition 3.1: The hybrid block method (6) is said to be consistent if it has an order more than or equal to one i.e. $P \geq 1$ [11].

3.3 Zero stability

Definition 3.2: The hybrid block method (6) said to be zero stable if the first characteristic polynomial $\pi(r)$ having roots such that $|r_z| \leq 1$ *and if* $|r_z| = 1$, then the multiplicity of r_z must not greater than two [11]. In order to find the zero-stability of hybrid block method (6), we only consider the first characteristic polynomial of the method according to definition (3.2) as follows

$$\Pi(r) = \left| r\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| = r^3(z-1)$$

Which implies $r = 0, 0, 0, 1$. Hence the method is zero-stable since $|r_z| \leq 1$.

3.4 Convergence

Theorem (3.1): Consistency and zero stability are sufficient condition for linear multistep method to be convergent. As the method (6) is consistent and zero stable, it indicates that the method is convergent for all point [11].

3.5 Regions of Absolute Stability (RAS)

The absolute stability region of the new method is A-stable and was found according to Sabo et al. 2018 and [11].

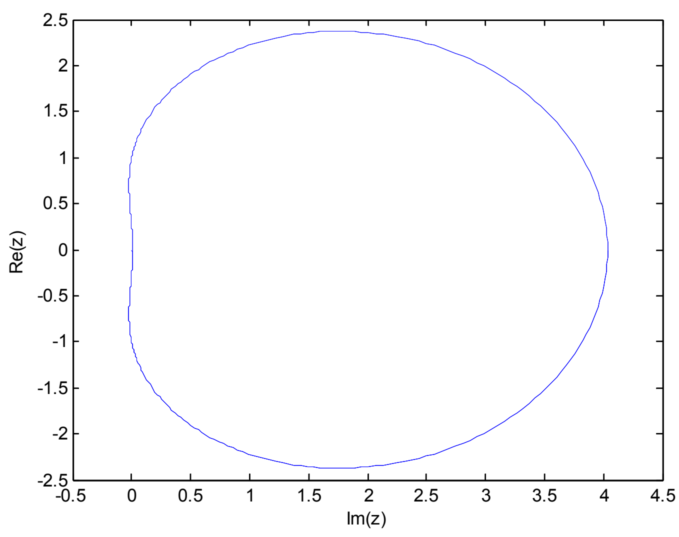

Fig. 1. Absolute stability of region new Enright methods

4 Numerical Implementation of new Enright's Method

The performance and the efficiency the new two-step Enright's linear multistep methods were tested on three test problems. The paper presents some numerical experiments widely solved by Althemai et al. [9]

and [2]. The performance of the new Enright method is examined on three systems of second-order stiffly initial value problems of ordinary differential equations. Tables 4.1, 4.2 and 4.3 show the comparison of our method with the other existing method [9] and [12], for absolute errors.

Definition 4.1: In mathematics, a stiff equation is a differential equation for which certain numerical method for solving the equation are numerical unstable, unless the step size is taking to be extremely small, (Dahlquist, 1963).

Problem 4.1

Consider the stiff system

$$y_1^1 = -8y_1 + 7y_2 ; y_1(0) = 1,$$
$$y_2^1 = 42y_1 - 43y_2 ; y_2(0) = 8, \quad h = \frac{1}{10}$$. With Exact Solution

$$y_1(x) = 2e^{-x} - e^{-50x}$$
$$y_2(x) = 2e^{-x} - 6e^{-50x}$$

Source, [9]

Table 4.1. Comparison of new Enright method with that of [9]

$X - value$	**Error in [9]** $K = 3$		**Error in New method** $K = 2$	
	$y_1(x)$	$y_2(x)$	$y_1(x)$	$y_2(x)$
0.1	1.38×10^{0}	3.20×10^{0}	3.87×10^{-4}	8.32×10^{-2}
0.2	9.02×10^{-1}	7.36×10^{-1}	6.00×10^{-6}	5.81×10^{-4}
0.3	1.09×10^{0}	2.58×10^{0}	6.90×10^{-8}	5.33×10^{-6}
0.4	9.09×10^{-1}	5.32×10^{0}	1.22×10^{-7}	1.36×10^{-7}
0.5	8.84×10^{-1}	2.10×10^{0}	4.50×10^{-8}	1.12×10^{-6}
0.6	7.22×10^{-1}	3.75×10^{0}	1.48×10^{-7}	1.36×10^{-7}
0.7	7.15×10^{-1}	1.71×10^{0}	8.08×10^{-8}	9.66×10^{-7}
0.8	6.42×10^{-1}	2.57×10^{0}	1.62×10^{-7}	1.52×10^{-7}
0.9	5.78×10^{-1}	1.39×10^{0}	1.03×10^{-7}	8.25×10^{-7}
1.0	5.68×10^{-1}	1.67×10^{-1}	1.66×10^{-7}	1.58×10^{-7}

Problem 4.2

Consider the stiff system

$$y_1' = 998y_1 + 1998y_2 \quad y_1(0) = 1$$
$$y_2' = -999y_1 - 1999y_2 \quad y_2(0) = 0, \quad h = 0.1$$

With Exact Solution

$$y_1(x) = 2e^{-x} - e^{-1000x}$$
$$y_2(x) = -e^{-x} - e^{-1000x}, \quad x \in [0,1]$$

Source, [10]

Table 4.2. Comparison of new Enright that of method with [10]

$X - value$	[10], $K = 3$		Error in New method $K = 2$	
	$y_1(x)$	$y_2(x)$	$y_1(x)$	$y_2(x)$
0.1	5.82×10^{-2}	5.83×10^{-2}	3.04×10^{-3}	2.90×10^{-3}
0.2	4.02×10^{-3}	3.95×10^{-3}	5.96×10^{-3}	5.93×10^{-3}
0.3	9.17×10^{-3}	2.16×10^{-3}	3.04×10^{-4}	1.60×10^{-4}
0.4	7.13×10^{-4}	6.26×10^{-4}	1.11×10^{-4}	2.34×10^{-5}
0.5	1.20×10^{-4}	4.22×10^{-5}	2.71×10^{-4}	1.35×10^{-4}
0.6	2.28×10^{-4}	1.57×10^{-4}	9.44×10^{-5}	4.745×10^{-5}
0.7	1.60×10^{-4}	7.77×10^{-5}	2.50×10^{-5}	1.25×10^{-4}
0.8	1.52×10^{-4}	7.67×10^{-5}	1.03×10^{-4}	5.14×10^{-5}
0.9	1.32×10^{-4}	6.78×10^{05}	2.27×10^{-4}	1.14×10^{-4}
1.0	1.52×10^{-4}	7.60×10^{-5}	1.05×10^{-4}	5.25×10^{-6}

Problem 4.3 Consider the stiff system

$$y_1' = 198y_1 + 199y_2 \quad y_1(0) = 1$$
$$y_2' = -398y_1 - 399y_2 \quad y_2(0) = -1, \quad h = 0.1$$

With Exact Solution

$$y_1(x) = e^{-x}$$
$$y_2(x) = -e^{-x}$$
$$x \in [0,1]$$

Source, [10]

Table 4.3. Comparison of new Enright method with that of [10],

$X - value$	Error in [10], $K = 3$		Error in New method $K = 2$	
	$y_1(x)$	$y_2(x)$	$y_1(x)$	$y_2(x)$
0.1	2.60×10^{-6}	2.60×10^{-6}	2.50×10^{-6}	9.05×10^{-6}
0.2	2.42×10^{-6}	2.42×10^{-6}	2.95×10^{-7}	2.81×10^{-7}
0.3	1.18×10^{-6}	1.18×10^{-6}	4.84×10^{-7}	4.33×10^{-6}
0.4	3.90×10^{-6}	3.90×10^{-6}	4.95×10^{-6}	4.83×10^{-6}
0.5	5.58×10^{-6}	5.58×10^{-6}	6.25×10^{-6}	5.84×10^{-6}
0.6	3.23×10^{-6}	3.23×10^{-6}	6.12×10^{-6}	6.02×10^{-6}
0.7	4.35×10^{-6}	4.35×10^{-6}	6.99×10^{-6}	6.66×10^{-6}
0.8	3.97×10^{-6}	3.97×10^{-6}	6.71×10^{-6}	6.63×10^{-6}
0.9	3.59×10^{-6}	3.59×10^{-6}	7.26×10^{-6}	6.98×10^{-6}
1.0	4.31×10^{-6}	4.30×10^{-6}	6.88×10^{-6}	6.82×10^{-6}

5 Conclusions

The formation of second derivative two-step block hybrid Enright's linear multistep methods to solve initial value problems of general second order stiff ordinary differential equations was studied. For this, we follow Enright's 1974 approach, by introducing the off-mesh points at both interpolation and collocations; we also developed the continuous schemes for new Enright's method. The analysis of the method was found to be convergent, consistent and zero-stable with absolute stable region. We further computed the order, error constants and plotted the region of absolute stability within which the method is A-stable. The absolute errors arising from Problems 4.1, 4.2 and 4.3 using the new Enright method were compared with the existing method [9] and [10]. It is evident from the results displayed in tables, 4.1, 4.2 and 4.3, that the newly derived Enright method performs better than the existing method [9] and [10] when implemented with numerical examples.

Competing Interests

Authors have declared that no competing interests exist.

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