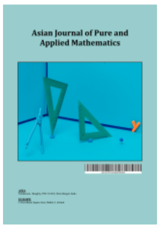

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Exponentially-Fitted One-Step Four Hybrid Point Methods for Solving Stiff and Oscillating Problems

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

The one-step, four hybrid point approach for solving second-order stiff and oscillatory differential equations is presented in this study. The continuous hybrid technique was created using the interpolation method and the collocation of the exponential function as the basis function. It was then evaluated at non-interpolating points to produce a continuous block method. When the continuous block was assessed at each stage, the discrete block approach was regained. Upon investigation, the fundamental characteristics of the techniques were discovered to be zero-stable, consistent, and convergent. The new method is used to solve a few stiff and oscillatory ordinary differential equation problems. Based on the numerical results, it was found that our approach provides a better approximation than the current method.

Keywords: One-step; hybrid point; second derivative; exponential fitted.

AMS subject classification: 65L05, 65L06, 65L20.

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1 Introduction

This study considers an approximate solution of second order ordinary differential equationsusing the one-step four off grid point hybrid approach of the type :

$$y'' = f\left(x, y, y'\right), \quad y\left(t_0\right) = y_0, \quad y'\left(t_0\right) = y'_0, \qquad (1)$$

The analytic solutions of second order ordinary differential equations can be found in a variety of ways. Equation (1) is of particular interest to researchers due to its broad range of applications in a variety of fields, including control theory, fluid dynamics, mathematical systems without dissipation, modeling scientific and engineering, celestial mechanics, and so on. There are more functions to evaluate when solving higher order derivatives using a first-order approach, which increases the computing load as seen in [1, 2]. For the solution of (1), several approaches, including hybrid and predictor-corrector approaches, have been put forth. Although predictor-corrector systems have shown promise, a significant drawback is that in predictor-corrector the accuracy tends to decrease, particularly at large step-length values and when results are observed at overlapping intervals.

Nonetheless, these shortcomings have been the subject of numerous studies [3,4,5,6]. The literature reports that the direct techniques of solving (1) yield higher accuracy, speed and are more efficient than the reduction method.

Among the scholars who have recently embraced the hybrid approach in lieu of the direct method for approximating (1) are [7,8,9,10,11].

The one-step, four-offgrid hybrid point method we devised in this research and implemented in block allows for the direct solution of second-order stiff and oscillatory problems.

The structure of the paper is as follows: Section 2 covers the materials and techniques used in the method's development. In Section 3, the method's basis properties are analyzed, numerical experiments are conducted to test the developed method's efficiency on a few numerical examples, and the findings are discussed. Finally, we wrapped up in section 4.

2 Derivation of the Method

This section describes the collocation approach by using the exponential function as the approximate solution and the objective of which the derivations of the method are used to obtain the algorithm in the form

$$y\left(t\right) = \tau_n\left(t\right) y_{n+i} + \tau_{n+i}\left(t\right) y_{n+i} + h^2 \left[\sum_{j=0}^{1} \psi_j\left(t\right) g_{n+j} + \sum_{i=\frac{1}{5}}^{\frac{4}{5}} \psi_i\left(t\right) g_{n+i} \right], \qquad (2)$$

$\tau_i(t)$, $\psi_j(t)$ are constant to be determine and τ_0 *and* ψ_0 are non zerohere.

Equation (2) is obtained by considering the exponential function as the basis approximate solution of the form

$$y(t)=\sum_{j=0}^{s+d-1}\left(\tau_j(t)\left(\ell^x\right)^j\right) \tag{3}$$

$s=6$ and $d=2$ are the numbers of collocation and interpolation points, the second derivative of (3) gives

$$g(x,y,y')=\sum_{j=2}^{s+d-1}\left(\frac{\tau_j j(t)}{h^2(j-2)!}\left(\ell^x\right)^{j-2}\right) \tag{4}$$

The continuouos approximation is then constructed by imposing two conditions which are

$$\left.\begin{aligned} y_{n+j}&=y\left(x_{n+j}\right), \quad j=0,\frac{1}{5} \\ y''\left(x_{n+j}\right)&=g_{n+j}, \quad j=0,\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5},1 \end{aligned}\right\} \tag{5}$$

Collocating (4) at all points and interpolating (3) at $d=0$, $\frac{1}{5}$ result to the system of non linear equation of the form

$$XA=U \tag{6}$$

Which will then be evaluated through a matrix inversion algorithm in which the values of τ_i's *and* ψ_j's are determined. By the substitutions of the values of τ_i's *and* ψ_j's obtained into equation (3) gives a continuous hybrid linear multistep method of the form (2)

We then impose (5) on $y(t)$ in (3) and the coefficients $y_{n+j}, i=0,\frac{1}{5}$ and $g_{n+j}, j=0,\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5},1$ give

$$y_{n+t}=\tau_0(t)y_n+\tau_{\frac{1}{2}}(t)\,y_{n+\frac{1}{2}}+h^2\left[\sum_{j=0}^{1}\psi_j g_{n+j}\right], \quad j=0\left(\frac{1}{5}\right)1 \tag{7}$$

Where $t=\dfrac{x-x_{n+2}}{h}$, $\qquad \dfrac{dt}{dx}=\dfrac{1}{h}$

$$\psi_0=-\frac{1}{420}h^2\,\frac{-14-10-21r-14r-21u-14u-21v-14v-21w-14w-35ru-21ru-35rv-21rv-35rw-21rw-35uv-21uv-35uw-21uw-35vw-21vw-70ruv-70ruw-70rvw-70uvw-35ruv-35ruw-35rvw-35uvw-210ruvw-70ruvw}{ruvw}$$

$$\psi_u=\frac{1}{420}h^2\,\frac{14-10-21r-14r-21v-14v-21w-14w-35rv-35rw-35vw-21rv-21rw-21vw-70rvw-35rvw}{u(u-1)(u-w)(u-v)(u-r)}$$

$$\psi_v=-\frac{1}{420}h^2\,\frac{14-10-21r-14r-21u-14u-21w-14w-35ru-35rw-35uw-21ru-21rw-21uw-70ruw-35ruw}{v(v-1)(v-w)(v-u)(v-r)}$$

$$\psi_r=-\frac{1}{420}h^2\,\frac{14-10-21u-14u-21v-14v-21w-14w-35uv-35uw-35vw-21uv-21uw-21vw-70uvw-35uvw}{r(r-1)(r-w)(r-v)(r-u)}$$

$$\psi_w=\frac{1}{420}h^2\,\frac{14-10-21r-14r-21u-14u-21v-14v-35ru-35rv-35uv-21ru-21rv-21uv-70ruv-35ruv}{w(w-1)(w-r)(w-u)(w-v)}$$

$$\psi_1=\frac{1}{420}h^2\,\frac{10-14r-14u-14v-14w-21ru-21rv-21rw-21uv-21uw-21vw-35ruv-35ruw-35rvw-35uvw-70ruvw}{(1-r)(1-u)(1-v)(1-w)}$$

The first derivative of (7) gives

$$y'(t)=\frac{1}{h}\left(\tau'_0(t)y_n+\tau'_{\frac{1}{2}}(t)y_{n+\frac{1}{2}}+h^2\left[\sum_{j=0}^{1}\psi'_j g_{n+j}+\sum_{i=\frac{1}{5}}^{\frac{4}{5}}\psi'_{S_i} g_{n+i}\right]\right) \tag{8}$$

We evaluate (7) at $t=t_{n+r},\ t=t_{n+v}, t=t_{n+w}, t=t_{n+1}$, *where* $u=\dfrac{1}{5}, v=\dfrac{2}{5}, r=\dfrac{3}{5}$ *and* $w=\dfrac{4}{5}$ to obtain

$$y_{n+\frac{2}{5}} = -y_n + 2y_{n+\frac{1}{5}} + \frac{3}{1000}h^2 f_n + \frac{209}{6000}h^2 f_{n+\frac{1}{5}} + \frac{1}{1500}h^2 f_{n+\frac{2}{5}} + \frac{7}{3000}h^2 f_{n+\frac{3}{5}} - \frac{1}{1000}h^2 f_{n+\frac{4}{5}} + \frac{1}{6000}h^2 f_{n+1}$$

$$y_{n+\frac{3}{5}} = -2y_n + 3y_{n+\frac{1}{5}} + \frac{7}{1200}h^2 f_n + \frac{221}{3000}h^2 f_{n+\frac{1}{5}} + \frac{101}{3000}h^2 f_{n+\frac{2}{5}} + \frac{13}{1500}h^2 f_{n+\frac{3}{5}} - \frac{13}{6000}h^2 f_{n+\frac{4}{5}} + \frac{1}{3000}h^2 f_{n+1}$$

$$y_{n+\frac{4}{5}} = -3y_n + 4y_{n+\frac{1}{5}} + \frac{13}{1500}h^2 f_n + \frac{337}{3000}h^2 f_{n+\frac{1}{5}} + \frac{53}{750}h^2 f_{n+\frac{2}{5}} + \frac{71}{1500}h^2 f_{n+\frac{3}{5}} + \frac{1}{1500}h^2 f_{n+\frac{4}{5}} + \frac{1}{3000}h^2 f_{n+1}$$

$$y_{n+1} = -4y_n + 5y_{n+\frac{1}{5}} + \frac{7}{600}h^2 f_n + \frac{3}{20}h^2 f_{n+\frac{1}{5}} + \frac{11}{100}h^2 f_{n+\frac{2}{5}} + \frac{13}{150}h^2 f_{n+\frac{3}{5}} + \frac{23}{600}h^2 f_{n+\frac{4}{5}} + \frac{1}{300}h^2 f_{n+1}$$

evaluating (8) at all points and simplifying gives the discrete hybrid block method of the form

$$A^{(0)} Y_m^{(i)} = \sum_{j=0}^{1} h^{(i)} e_i y_n^{(i)} + h^2 b_i g(y_n) + h^2 c_i g(y_m) \qquad (9)$$

Where $A^{(0)} = 5\times 5$ *identity matrix*

$$Y_m^{(i)} = \left[y_{n+\frac{1}{5}}, y_{n+\frac{2}{5}}, y_{n+\frac{3}{5}}, y_{n+\frac{4}{5}}, y_{n+1}\right]^T, \quad y_n^i = \left[y_{n-\frac{2}{5}}, y_{n-\frac{3}{5}}, y_{n-\frac{4}{5}}, y_{n-1}, y_n\right]^T$$

$$g(y_m) = \left[g_{n+\frac{1}{5}}, g_{n+\frac{2}{5}}, g_{n+\frac{3}{5}}, g_{n+\frac{4}{5}}, g_{n+1}\right]^T, \quad g(y_n) = \left[g_{n-\frac{2}{5}}, g_{n-\frac{3}{5}}, g_{n-\frac{4}{5}}, g_{n-1}, g_n\right]^T$$

when $i = 0, 1$ and $u = \frac{1}{5}$, $v = \frac{2}{5}$, $r = \frac{3}{5}$ *and* $w = \frac{4}{5}$

We obtain the following discrete scheme

	y_n	y'_n	g_n	$g_{n+\frac{1}{5}}$	$g_{n+\frac{2}{5}}$	$g_{n+\frac{3}{5}}$	$g_{n+\frac{4}{5}}$	g_{n+1}
$y_{n+\frac{1}{5}}$	1	$\frac{h}{5}$	$\frac{1231}{126000}h^2$	$\frac{863}{50400}h^2$	$-\frac{761}{63000}h^2$	$\frac{941}{126000}h^2$	$-\frac{341}{126000}h^2$	$\frac{107}{252000}h^2$

$y_{n+\frac{2}{5}}$	1	$\frac{2h}{5}$	$\frac{71}{3150}h^2$	$\frac{544}{7875}h^2$	$-\frac{37}{1575}h^2$	$\frac{136}{7875}h^2$	$-\frac{101}{15750}h^2$	$\frac{8}{7875}h^2$
$y_{n+\frac{3}{5}}$	1	$\frac{3h}{5}$	$\frac{123}{3500}h^2$	$\frac{3501}{28000}h^2$	$-\frac{9}{3500}h^2$	$\frac{87}{2800}h^2$	$-\frac{9}{875}h^2$	$\frac{9}{5600}h^2$
$y_{n+\frac{4}{5}}$	1	$\frac{4h}{5}$	$\frac{376}{7875}h^2$	$\frac{1424}{7875}h^2$	$\frac{176}{7875}h^2$	$\frac{608}{7875}h^2$	$-\frac{16}{1575}h^2$	$\frac{16}{7875}h^2$
y_{n+1}	1	h	$\frac{61}{1008}h^2$	$\frac{475}{2016}h^2$	$\frac{25}{504}h^2$	$\frac{125}{1008}h^2$	$\frac{25}{1008}h^2$	$\frac{11}{2016}h^2$
$y'_{n+\frac{1}{5}}$		1	$\frac{19}{288}h$	$\frac{1427}{7200}h$	$-\frac{133}{1200}h$	$\frac{241}{3600}h$	$-\frac{173}{7200}h$	$\frac{3}{800}h$
$y'_{n+\frac{2}{5}}$		1	$\frac{14}{255}h$	$\frac{43}{150}h$	$\frac{7}{225}h$	$\frac{7}{225}h$	$-\frac{1}{75}h$	$\frac{1}{450}h$
$y'_{n+\frac{3}{5}}$		1	$\frac{51}{800}h$	$\frac{219}{800}h$	$\frac{57}{400}h$	$\frac{57}{400}h$	$-\frac{21}{800}h$	$\frac{3}{800}h$
$y'_{n+\frac{4}{5}}$		1	$\frac{14}{225}h$	$\frac{64}{225}h$	$\frac{8}{75}h$	$\frac{64}{225}h$	$\frac{14}{225}h$	0
y'_{n+1}		1	$\frac{19}{288}h$	$\frac{25}{96}h$	$\frac{25}{144}h$	$\frac{25}{144}h$	$\frac{25}{96}h$	$\frac{19}{288}h$

(10)

3 Analysis of Basic Properties of the Method

3.1 Order of the Block

According to fatunla (1991) and lambert (1973) the truncation error associated with (2) is defined by

$$L\left[y(x);h\right]=\tau_0(t)y_n+\tau_{\frac{1}{5}}(t)y_{n+\frac{1}{5}}+h^2\left[\sum_{j=0}^{1}\psi_j(t)g_{n+j}+\sum_{i=\frac{1}{5}}^{\frac{4}{5}}\psi_i(t)g_{n+i}\right] \tag{11}$$

Assumed that $y(x)$ can be differentiated. Expanding (10) in Taylor's series and comparing the coefficient of h gives the expression

$$L\{y(x):h\}=C_0y(x)+C_1y'(x)+,...+C_ph^py^p(x)+C_{p+1}h^{p+1}y^{p+1}(x)$$

Where the constant coefficients are given below

$$C_0=\sum_{j=0}^{k}\tau_j\ ,\quad C_1=\sum_{j=1}^{k}j\tau_j$$

$$C_q=\frac{1}{q!}\sum_{j=0}^{k}j^q\tau_j-q(q-1)\left(\sum_{j=0}j^{q-2}\psi_j+\left(\frac{1}{5}\right)^{q-2}\psi_{\frac{1}{5}}+\left(\frac{2}{5}\right)^{q-2}\psi_{\frac{2}{5}}+\left(\frac{3}{5}\right)^{q-2}\psi_{\frac{3}{5}}+\left(\frac{4}{5}\right)^{q-2}\psi_{\frac{4}{5}}+1^{q-2}\psi_1\right),$$

$q=2,3,4...$

Definition 1: the linear operator and the associated continuous linear multistep method (11) are said to be of order p if $c_0=c_1=c_2=\ldots=c_p=c_{p+1}=0,\ and\ c_{p+2}\neq 0$, c_{p+2} is called the error constant and the local truncation error is given by

$$t_{n+k}=c_{p+2}h^{(p+2)}y^{(p+2)}(x_n)+o\left(h^{p+3}\right) \qquad (12)$$

In our method

Comparing the coefficient of h gives $C_0=C_1=C_2=C_3=\ .\ .\ .\ =C_7=0$ and

$$C_8=\left[-\frac{863}{787\,500\,000},-\frac{37}{49\,218\,750},-\frac{87}{87\,500\,000},-\frac{16}{24\,609\,375},-\frac{11}{6300\,000}\right]^T$$

Hence our method is of order five (5).

3.2 Consistency

The One-step Hybrid exponentially fitted second derivative is consistent since it has order greater than or equal to one.

3.3 Zero stability of our method

The One-Step four Hybrid points Block exponentially fitted second derivative method is said to be zero-stable if as $h\to 0$, the root $z_i, i=0\left(\frac{1}{5}\right)1$ of the first characteristic polynomial $\rho(z)=0$, that is $\rho(z)=\det\left[\sum_{j=0}^{k}A^{(i)}z^{k-i}\right]=0$ Satisfies $|z_i|\leq 1$ and for those roots with $|z_i|=1$, multiplicity must not exceed two.

3.4 Convergency

The compulsory terminolgy for the exponential fitted to be convergent is that they must be consistent and zero-stable. Hence, our method converges since all conditions are satisfied.

3.5 Linear stability

According to Hairer and Wanner, the concept of A-satbility is discussed by applying the test equation

$$y^{(k)}=\lambda^{(k)}y$$

to yield

$$Y_m=M(z)Y_{m-1},\ z=\lambda h$$

where is the amplification matrix given by

$$M(z) = \left(A^0 - zB^0 - z^2C^0\right)^{-1}\left(A^1 - zB^1 - z^2C^1\right)$$

The matrix $M(z)$ has eigen values $\left(0,\,0,\,.\,.\,.\,.\,,\,\xi_k\right)$ where ξ_k is called the stability function. Thus, ths stability function of our method with four off-grid points is given by

$$\xi = \frac{\left(12z^5 + 274z^4 + 3375z^3 + 25\,500z^2 + 112\,500z + 225\,000\right)}{12z^5 - 274z^4 + 3375z^3 - 25\,500z^2 + 112\,500z - 225\,000}$$

3.6 Region of absolute stability

The stability polynomial of our method is found to be

$$-\frac{1}{9375}h^5 + h^4\left(\frac{149}{225\,000}w - \frac{149}{225\,000}\right) - h^3\left(\frac{229}{10\,800}w^2 + \frac{19}{2160}w\right) - h^2\left(\frac{1873}{21\,600}w^2 - \frac{1873}{21\,600}w^3\right) - h\left(\frac{163}{288}w^4 + \frac{125}{288}w^3\right) + w^5 - w^4$$

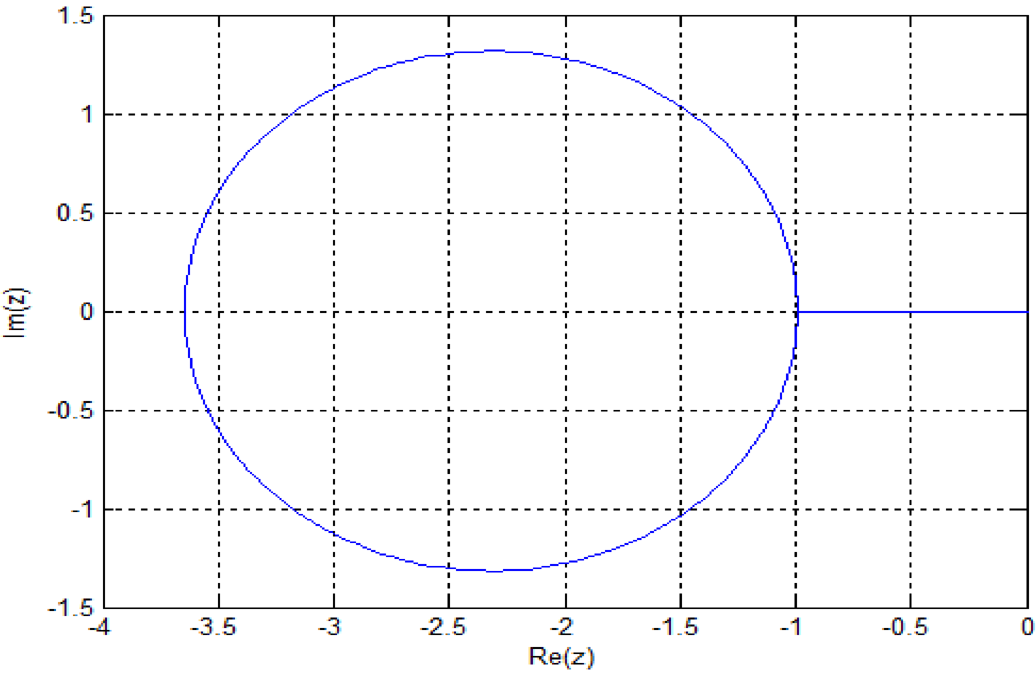

Fig. 1. The stability Region of our method

3.7 Mathematical Computation of the method

Problem I We consider the stiff equation (Source: Adeniran et al. [5])

$$y'' = -1001y' - 1000y \quad, y(0) = 1\ ,\ \ y'(0) = -1 \quad h = 0.1$$

Exact Solution: $y(x) = e^{-x}$,

Table 1. Comparison of the proposed method with Adeniran et al. [5]

x-values	Exact Solution	Computed Solution	Error in our method	Error in [6]
0.1	0.90483741803595957316	0.90483741803595977302	1.99860e-16	1.689E-11
0.2	0.81873075307798185867	0.81873075307798198612	1.27450e-16	1.418 E-11
0.3	0.74081822068171786607	0.74081822068171808378	2.17710e-16	1.627E-11
0.4	0.67032004603563930074	0.67032004603563949348	1.92740e-16	1.663E-11
0.5	0.60653065971263342360	0.60653065971263365646	2.32860e-16	1.710E-11
0.6	0.54881163609402643263	0.54881163609402665616	2.23530e-16	1.725E-11
0.7	0.49658530379140951470	0.49658530379140975434	2.39640e-16	1.723 E-11
0.8	0.44932896411722159143	0.44932896411722182552	2.34090e-16	1.706 E-11
0.0	0.40656965974059911188	0.40656965974059934996	2.38080e-16	1.676E-11
1.0	0.36787944117144232160	0.36787944117144255393	2.32330e-16	1.637E-11

Problem II Consider the highly Oscilatory equation (source: Adeniran and Edaogbogun (2021))

$$y'' = -\lambda^2 y, \quad y(0)=1,\ y'(0)=2,\ \lambda=2 \ \ ,\ \ h=0.01$$

Exact Solution: $y(x)=\cos 2x+\sin 2x$

Table 2. Comparison of the proposed method with Adeniran and Edaogbogun [5]

x-values	Exact Solution	Computed Solution	Error in our method	Error in [5]
0.1	1.0197986733599108578	1.0197986733599108578	0.00e+00	4.3881E-11
0.2	1.0391894408476120998	1.0391894408476120998	0.00e+00	7.9019E-11
0.3	1.0581645464146487647	1.0581645464146487647	0.00e+00	2.5525E-10
0.4	1.0767164002717920723	1.0767164002717920723	0.00e+00	1.1525E-10
0.5	1.0948375819248539184	1.0948375819248539184	0.00e+00	1.9079E-10
0.6	1.1125208431427856122	1.1125208431427856122	0.00e+00	2.3002E-10
0.7	1.1297591108568736536	1.1297591108568736537	1.10e-19	2.7014E-10
0.8	1.1465454899898729124	1.1465454899898729125	1.10e-19	3.1112E-10
0.9	1.1628732662139455929	1.1628732662139455932	3.10e-19	3.5291E-10
1.0	1.1787359086363028466	1.1787359086363028469	3.10e-19	3.9545E-10

Problem III The temperature degrees of a body, minutes after being placed in a certain room, satisfy the differential equation $3\frac{d^2y}{dt^2}+\frac{dy}{dt}=0$. By using the substitution $\frac{dy}{dt}=x$ or otherwise, find y in terms of t given that y=60 when t=0, y=35 and t=6In4. Find out after how many minutes the rate of cooling of the body will have fallen below one degree per minute, giving your answer correct to the nearest minute. The problem is mathematically modeled as follows:

$$y''(t) = -\frac{y'(t)}{3}\ ,\ y(0)=60\ ,\ y'(0) = -\frac{80}{9}$$

Exact Solution: $y(t)=\frac{80}{3}e^{-\left(\frac{1}{3}\right)t}+\frac{100}{3}$

Table 3. Comparison of the suggested method

x-values	Exact Solution	Computed Solution	Error in our method	Error in [3]
0.1	59.12576267952015738700	59.12576267952015739300	6.00000e-18	9.245422e-16
0.2	58.28018626750980633900	58.28018626750980636200	2.30000e-17	7.891086e-16
0.3	57.46233114762558861800	57.46233114762558866700	4.90000e-17	5.926176e-16
0.4	56.67128850781193210600	56.67128850781193219200	8.60000e-17	4.684342e-15
0.5	55.90617933041637530800	55.90617933041637543900	1.31000e-16	3.123519e-15
0.6	55.16615341541284956400	55.16615341541284974900	1.85000e-16	4.647865e-14
0.7	54.45038843564751105000	54.45038843564751129700	2.47000e-16	3.261193e-14
0.8	53.75808902305729847200	53.75808902305729878800	3.16000e-16	2.845575e-14
0.9	53.08848588484580976200	53.08848588484581015400	3.92000e-16	1.167002e-14
1.0	52.44083494863438001100	52.44083494863438048500	4.74000e-16	3.639360e-13

4 Conclusions

It is evident from the above tables that our proposed method has significant improvement over the existing methods. The One-step four hybrid point exponentially fitted method is proposed for direct solution of general second order stiff and oscillatory problems where by it is self-starting when implemented. The developed method converges and it is of order five.

Competing Interests

Authors have declared that no competing interests exist.

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