



# **Boolean Subtraction and Division with Application in the Design of Digital Circuits**

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## **Author's contribution**

*The sole author designed, analysed, interpreted and prepared the manuscript.*

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## **ABSTRACT**

The subject of Boolean subtraction and division dates back for over a century to the work of George Boole. Nevertheless, this subject is unfamiliar to us because it has been banished from Boolean algebra. In fact, some authors claim that there is no such thing as Boolean subtraction and division. The purpose of this work, however, is to present with clarity the subject of logical subtraction and division and its practical application in the design of digital circuits.

*Keywords: logic zero; logic one; Boolean subtraction; Boolean division; Bhaskarization methods; digital circuits.*

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## **1 INTRODUCTION**

In the days of Buddha in India in the fifth century BCE, the venerable principle of reasoning called the Catuskoti was renowned because of certain

questions the buddists were asking Buddha. The Catuskoti insists that there are four answer possibilities or options regarding such questions: it might be true, false, both true and false, and neither true nor false [1]. At almost the same time

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of Buddha, Aristotle was laying the groundwork of logical reasoning in the West along different pathway. He enunciated the principle of excluded middle which asserts that every proposition must be either a true statement or a false one; there are no other option [2].

In his *The Investigation of the Laws of Thought* Boole creates his algebra of logic called Boolean algebra and defines Boolean addition and multiplication, and then speaks of Boolean subtraction as reversing addition and of Boolean division as undoing multiplication:

(...) we cannot conceive it possible to collect parts into a whole, and not conceive it also possible to separate a part from a whole.

Peirce sees Boolean subtraction and division as having no interpretation and modifies Boole's system by banishing these operations and they have not been made use of since [3], [4].

In 1910 Ehrenfest points out the possibility of applying Boolean algebra in the analysis of electrical circuits containing switches [5]:

Is it right, that regardless of the existence of the already elaborated algebra of logic, the specific algebra of switching networks should be considered as a utopia?

From 1934 to 1936, Nakashima constructs his switching algebra based upon the fact that a relay contact has impedance that is a function of time whose value shall be limited to either zero or infinite. If  $A$  and  $B$  are two-terminal circuits, which are called simple partial paths, then  $A + B$  and  $A \times B$  correspond to their serial and parallel connections, respectively [6]. Shannon, in his paper [7] uses 0 to represent the zero impedance of a closed switch and 1 to indicate the infinite impedance of an open switch, thereby making the simplification of combined switches easier to handle. This is the significance of Shannon's contribution [8].

With the advent of electronic devices and computers, Boolean algebra has become prominent means for analyzing and designing digital circuits [9]. So many works on it have appeared before the public presenting in clear and simple language various aspects of the

subject of the Algebra [10], [11], [12]. The vast majority of these entirely ignore the theme of logical subtraction and division [9], [13], and in consequence we are in almost total ignorance concerning those things which pertain to logical subtraction and division. It is for this reason this article has been prepared. Thus, this work is calculated to awaken a deep interest in logical subtraction and division and lead to definite re-investigations.

The remainder of this work is divided into four sections. The second section is devoted entirely to the basic concepts of Boole's algebra. Section 3 deals with the Bhaskarization of some Boolean expressions into expressions with logical interpretations. Section 4 treats of methods of Bhaskarizing Boolean expressions, hopefully in a sufficiently didactic manner that the last section on the applications of Boolean subtraction and division to design of digital circuits can be easily comprehended.

It is expected that the reader is familiar with Boolean addition and multiplication and their applications in the design of digital circuits.

## 2 BASIC CONCEPTS

### 2.1 Logical Values

The *logical values* are the values of the two possible states of any Boolean quantity. They are the stating values and expected results of every Boolean operations. They are 0 (zero) and 1 (unity) and their respective interpretations are empty set and universal set in the algebra of set.

### 2.2 Boolean Addition and Multiplication

**Definition 2.1.** Let  $A$  and  $B$  be Boolean variables. The Boolean variable  $A + B$  is another Boolean variable which equals 0 when both  $A$  and  $B$  equal 0 and equal 1 otherwise. The variable  $A + B$  is called the Boolean sum of  $A$  and  $B$ . The Boolean variable  $A \times B$  or simply  $AB$ , is another Boolean variable which equals 1 when both  $A$  and  $B$  equal 1 and equal 0 otherwise. The variable  $AB$  is called the Boolean product of  $A$  and  $B$ .

The truth table for  $A + B$  is shown in Table 1.

**Table 1. Truth Table for  $A + B$**

$A$	$B$	$A + B$
0	0	0
0	1	1
1	0	1
1	1	1

The truth table for  $AB$  is shown in Table 2.

**Table 2. Truth Table for  $AB$**

$A$	$B$	$AB$
0	0	0
0	1	0
1	0	0
1	1	1

## 2.3 Boolean Subtraction and Division

Boolean addition and multiplication are very clear. Now we must go some steps further. We gain a complete understanding of Boolean algebra by considering Boolean subtraction and division. It is impossible to be satisfied with the Boolean operations of addition and multiplication alone. The operations of subtraction and division are noteworthy since they produce illogical values alongside with the logical values. Passing from Boolean addition and multiplication to Boolean subtraction and division respectively, therefore, brings new difficulties. We must have the courage to overcome them if we wish to understand the original algebra of Boole. Our endeavour, therefore, will be to unravel the notions of Boolean subtraction and division as they appear in Boole's work and even beyond it. We shall try to go at once to the heart of the matter and grasp the real significance of Boolean subtraction and division.

### 2.3.1 Boolean Subtraction

The Boolean subtraction, whose operator is denoted by  $-$ , acquires its existence from Boolean addition. Theorem 2.1 provides the results of Boolean subtraction for the possible combinations of 0 and 1.

**Theorem 2.1.** *Let  $a$  and  $b$  be logical values. Then*

$$a - b = \begin{cases} 0 & \text{if } a=0 \text{ and } b=0, \\ 1 & \text{if } a=1 \text{ and } b=0, \\ -1 & \text{if } a=0 \text{ and } b=1, \\ \{0, 1\} & \text{if } a=1 \text{ and } b=1. \end{cases}$$

*Proof.* Let  $a - b = X$  where  $X$  is the Boolean difference of  $a$  and  $b$ . Then expressing  $a$  in terms of  $X$  and  $b$  results in the equation

$$a = X + b. \tag{2.1}$$

Case I. If  $a = 0$  and  $b = 0$ , we get

$$0 = X + 0.$$

In Table 1 we have only one case in which the Boolean sum is 0, namely,  $0 = 0 + 0$ . From this we infer that the equation  $0 = X + 0$  is valid only if  $X = 0$ . Assigning logic 1 to  $X$  renders the equation invalid. This proves the first case.

Case II. If  $a = 1$  and  $b = 0$ , we get

$$1 = X + 0.$$

In Table 1 we have only one case in which the Boolean sum is 1, namely,  $1 = 1 + 0$ . From this we infer that the equation  $1 = X + 0$  is valid if  $X = 1$ . Assigning logic 0 to  $X$  renders the equation unsatisfied. This proves the second case.

Case III. If  $a = 0$  and  $b = 1$ , we obtain

$$0 = X + 1. \tag{2.2}$$

If  $X$  is assigned logic 0 or 1, the summation expression  $X + 1$  will always give logic 1. Thus,  $X$  takes neither 0 nor 1. Let us rewrite (2.2) as

$$0 - 1 = X.$$

We rewrite this last equation as

$$-(1 - 0) = X.$$

From Case II we obtain

$$-1 = X$$

and the proof of Case III is finished.

Case IV. If  $a = 1$  and  $b = 1$ , we obtain

$$1 = X + 1. \tag{2.3}$$

From Table 1 we see that this equation holds if  $X$  is assigned either logic 0 or 1 and this completes the proof of Case IV.  $\square$

Our results are summarized in the truth table 3 which defines  $A - B$ .

**Table 3. Truth Table for  $A - B$**

$A$	$B$	$A - B$
0	0	0
0	1	-1
1	0	1
1	1	{0, 1}

### 2.3.2 Boolean Division

We now have to take the last operation, namely, Boolean division. What is Boolean division? The operation, denoted by  $/$ , arises from the attempt to reverse Boolean multiplication. Its quotients, called Boolean quotients, are given by Theorem 2.2.

**Theorem 2.2.** *Let  $a$  and  $b$  be fixed logical values. Then*

$$\frac{a}{b} = \begin{cases} 10 & \text{if } a=1 \text{ and } b=1, \\ 0 & \text{if } a=0 \text{ and } b=1, \\ \frac{1}{0} & \text{if } a=1 \text{ and } b=0, \\ \{0, 1\} & \text{if } a=0 \text{ and } b=0. \end{cases}$$

*Proof.* Let  $a/b = X$  where  $X$  is the logical quotient of  $a$  and  $b$ . Then expressing  $a$  in terms of  $X$  and  $b$  results in the equation

$$a = X \times b. \tag{2.4}$$

Case I. If  $a = 1$  and  $b=1$ , we get

$$1 = X \times 1.$$

From Table 2 this equation is valid if  $X = 1$ . Assigning logic 0 to  $X$  renders the equation invalid.

Case II. If  $a = 0$  and  $b = 1$ , we get

$$0 = X \times 1.$$

From Table 2 this equation is valid if  $X = 0$ . Putting logic 1 in place of  $X$  invalidates the equation.

Case III. If  $a = 1$  and  $b = 0$ , we obtain

$$1 = X \times 0. \tag{2.5}$$

This equation is invalid, for if  $X$  is assigned logic 0 or 1, the multiplication expression  $X \times 0$  will always give logic 0. Thus,  $X$  cannot be a logical value.

We re-frame eq. (2.5) as

$$\frac{1}{0} = X.$$

and call  $1/0$  logic infinity.

Case IV. If  $a = 0$  and  $b = 0$ , we obtain

$$0 = X \times 0. \tag{2.6}$$

From Table 2. we notice that eq. (2.6) holds good if  $X$  is assigned either logic 0 or 1 and this completes the proof of Case IV.  $\square$

We summarize our results on Boolean division in the truth table shown in table 4.

**Table 4. Truth Table for  $A/B$**

$A$	$B$	$A/B$
0	0	$\{0, 1\}$
0	1	0
1	0	$1/0$
1	1	1

## 2.4 Illogical Values and Laws of Boolean Signs

The *illogical values* are values which are not of the two possible states of a Boolean quantity. They depend on the logical values, 1 and 0, for their existence and “are of a nature altogether foreign to the province of general reasoning” [14]. They are  $-1$  and  $\frac{1}{0}$ . What makes them really useful is that we can calculate with them as with logical values.

The laws of logical signs are the same in Boolean Algebra as in Ordinary Algebra:

$$\begin{aligned} + \times + &= + \\ + \times - &= - \\ - \times - &= + \\ - \times + &= - \end{aligned}$$

and

$$\begin{aligned} + \div + &= + \\ + \div - &= - \\ - \div - &= + \\ - \div + &= - \end{aligned}$$

Hence, like signs produce plus and unlike signs minus.

## 2.5 Boolean Theorems

The Boolean theorems include the following.

**Theorem 2.3** (Annulment Law).  $1 + A = 1$  and  $A \times 0 = 0$ .

**Theorem 2.4** (Identity Law).  $0 + A = A$  and  $A \times 1 = A$ .

**Theorem 2.5** (Idempotent Law).  $A + A = A$  and  $A \times A = A$ .

**Theorem 2.6** (Double Negation Law).  $\overline{\overline{A}} = A$ .

**Theorem 2.7** (Complement Law).  $\bar{A} + A = 1$  and  $A \times \bar{A} = 0$ .

**Theorem 2.8** (Commutative Law).  $A + B = B + A$  and  $AB = BA$ .

**Theorem 2.9** (Associative Law).  $A + (B + C) = (A + B) + C = A + B + C$  and  $A(BC) = (AB)C = ABC$ .

**Theorem 2.10** (Distributive Law).  $A(B + C) = AB + AC$  and  $A + (BC) = (A + B)(A + C)$ .

**Theorem 2.11** (Absorptive Law).  $A + (AB) = A$  and  $A(A + B) = A$ .

**Theorem 2.12** (De Morgan's Theorem 1).  $\overline{A + B} = \bar{A}\bar{B}$ .

**Theorem 2.13** (De Morgan's Theorem 2).  $\overline{AB} = \bar{A} + \bar{B}$ .

Proofs of these theorems are well-known and may be found in [13], [8].

## 2.6 Simple and Complex Boolean Expressions

Boolean expressions can either be simple or complex depending on whether or not it can be simplified into simpler forms by means of the Boolean theorems already mentioned.

**Definition 2.2.** A simple Boolean expression is one that resists simplification to a simpler form. If an expression can be simplified to simpler logical form, it is said to be complex.

Thus, the expressions  $A$  and  $A + \bar{B}$  are simple expressions as they can never be reduced to simpler logical forms. On the other hand, the expressions  $A + \bar{A}B$  and  $A + AB$  are complex expressions because they can be reduced to simpler logical forms, viz

$$A + \bar{A}B = A + B$$

and

$$A + AB = A.$$

## 3 BHASKARIZATION OF BOOLEAN EXPRESSIONS

As the ultimate objects of Boolean subtraction and division are to reverse the processes of Boolean addition and multiplication respectively, the indeterminate forms obtained in the previous section, viz

$$1 - 1 = \{0, 1\}$$

$$\frac{0}{0} = \{0, 1\},$$

are of utmost importance in the consideration of Boolean algebra. These indeterminate forms will be better understood if we discuss that which we call *Bhaskara's principle of impending operation on zero*. In his *Lilavati*, Bhaskara II claims [15], [16], [17], [18], [19]:

When a number is multiplied by cipher, the product is cipher; but in case any operation remains to be done, cipher is considered to be the multiplier, and if cipher also becomes the divisor, the number is considered unchanged.

The reasoning here concedes nothing false and conveys no wrong impression. In essence Bhaskara is saying that

$$A \times 0 = 0 \tag{3.1}$$

where  $A$  is a finite number. Suppose, after the operation of simplifying  $A \times 0$  to 0, there is an approaching operation in which 0 is a divisor. In this case it would be

$$\frac{0}{0}$$

which is indeterminate, that is, the answer cannot be known from this ratio of zero to zero. But according to Bhaskara, we should return and use the known form  $A \times 0$  (and not merely 0 to which  $A \times 0$  equals) in this new operation such that when it is divided by 0, the result gives  $A$ , that is

$$\frac{A \times 0}{0} = A.$$

This leads us to the definition of division of zero by itself [19].

**Definition 3.1** (Bhaskara II). Let 0 be zero and  $A$  any real number. Then

$$\frac{A \times 0}{0} = A$$

where the multiplier 0 is also the divisor 0, i.e both zeros are identical with each other.

Genesa wrote a commentary *Buddhivilasini* (1545 CE) where he made this principle clearer. According to him, "When a quantity  $A$  has zero multiplier and another operation is there, then the rule  $A \times 0 = 0$  should not be applied. But the 0 should be placed at its side as a multiplier. If for the remaining operation, 0 is a divisor, then due to identical multiplier and divisor, the zero 0 (in the numerator and denominator) should be cancelled "[15]. In keeping with Genesa, the ratio of identical zeros equal unity on the ground that the two zeros, being the same in every respect, cancel out to give the number 1.

**Definition 3.2** (Genesa-Bhaskara II). Let 0 be zero and 1 unity. Then the ratio of 0 to itself is

$$\frac{0}{0} = 1.$$

Let us enter more deeply into this principle in order to reach a difference between what we call *identity* and the term *equivalence*. With this principle Bhaskara introduces the abstract idea that in further operations  $A \times 0$  need not be 0 but it must be retained as  $A \times 0$ , or otherwise  $A \times 0 = 0$ . From this it is implied that the "0" in the second member of  $A \times 0 = 0$  must be re-expressed as  $A \times 0$  in impending operations. From this also it may be inferred that the two zeros, the multiplier 0 and the product 0, are non-identical, for if this were not the case, the product 0 would not have been re-expressed as  $A \times 0$  in further operations.

What then is the relationship between these zeros which we are discussing? We answer, they are equivalent to each other. Though both are expressed with the same symbol, namely, 0, they are nevertheless of different origins. Since the expression  $A \times 0$  equals the "0" in the second member of  $A \times 0 = 0$ , it follows that the "0" in the second member is identical with  $A \times 0$ . If we assume the "0" in the first member of  $A \times 0 = 0$

originate from  $1 \times 0$  (called unit zero), then it follows that the "0" in the second member is from  $A \times 1 \times 0$ . From this we may deduce that the two zeros become identical only when  $A = 1$ .

Now, let us divide both members of eq. (3.1) by the unit zero, i.e the zero in the first member. Let us also suppose that  $A$  is a number apart from unity. The equation becomes

$$\frac{A \times 0}{0} = \frac{0}{0}.$$

In the first member, the 0 as a multiplier of the numerator and the 0 as the entire denominator are exactly the same or identical with each other and so, the first member reduces to  $A$ . Thus, we now have

$$A = \frac{0}{0}$$

that is, the ratio of 0 to 0 is the number  $A$  which is not unity. The ratio is not unity because the ratio is not of two identical zeros as the numerator 0 is derived from the product of the unit zero and the number  $A$  which we have assumed to be different from unity. It is, however, impossible to reach  $A$  by merely considering the symbol  $0/0$ . It is in order to overcome the indeterminacy of the ratio of two equivalent zeros that the genius enunciated his principle of impending operation on zero. The process of returning to a previous or known operation in order to obtain the actual result of an impending operation leading to an indeterminate form may be termed Bhaskarization.

**Definition 3.3.** Bhaskarization is the art and process of returning to a previous operation or known operation in order to obtain the actual answer(s) to an impending operation leading to an indeterminate form.

Bhaskarization is important in Boolean algebra because both Boolean subtraction and division leads to the indeterminate forms as already mentioned, namely,  $1 - 1$  and  $\frac{0}{0}$ .

From the definition  $1/1 = 1$  and by definition 3.2 or taking the idea  $0/0 = 1$  where 0 is logic zero and 1 logic unity [20], the following important theorem of Boolean identity can be proved.

**Theorem 3.1.** Let  $A$  be a Boolean variable. Then

$$\frac{A}{A} = 1$$

where dividend  $A$  is also divisor  $A$ , i.e both  $A$  s are identical with each other.

A consequence of this theorem is the following theorem of identity.

**Theorem 3.2.** Let  $A$  be a Boolean variable. Then

$$A - A = 0$$

where minuend  $A$  is also subtrahend  $A$ .

The proof of this is so simple that we ignore it. The reader can easily reach it from Theorem 3.1 following the approach of ordinary algebra.

If the two  $A$ 's in Theorem 3.1 and Theorem 3.2 are not identical, then their ratio cannot equal logic 1 and their difference cannot equal logic 0. Before we proceed further into this matter, let us make clear the difference between identical expressions and equivalent ones by way of definitions.

**Definition 3.4.** Two Boolean expressions are identical with each other if they are exactly equal to each other in every respect and detail. Two Boolean expressions are equivalent if they are virtually equal in some respect.

Equivalent expressions cannot be equated to each other because they are not equal to each other in every respect. Moreover, cancellation of a Boolean expression from both members of a Boolean equation involving equivalent Boolean products is not permissible.

Suppose  $AE_1 - AE_2 = 0$  which is the same as  $AE_1 = AE_2$  where the  $A$ s are identical and  $E_1$  and  $E_2$  are different simple Boolean expressions. Assume the expressions  $AE_1$  and  $AE_2$  are identical because they are equal to each other indicated by the equality sign. Since this is true, we rewrite the assumed equation as

$$\frac{AE_1}{AE_2} = 1;$$

unity in the second member arising because every expression is identical with itself. Cancelling out the the identical  $A$ s will result in the contradiction

$$\frac{E_1}{E_2} = 1.$$

Hence, the expressions  $AE_1$  and  $AE_2$  are not identical; they are mere equivalents.

As an instance, take the equation

$$A(A + \overline{B}) = A(A + B).$$

Dividing both sides by the second member gives

$$\frac{A(A + \overline{B})}{A(A + B)} = 1$$

In this last equation,  $A(A + \overline{B}) = A$  and  $A(A + B) = A$  so that the equation is satisfied, it does not follow that

$$\frac{A + \overline{B}}{A + B} = 1,$$

cancelling out the two identical multiplicands, the  $A$ s.

Because equivalent complex expressions are not equal in every respect, the simple expressions to which they can be reduced are not identical but mere equivalent even though these simple expressions are in the same form. For instance, the two  $A$ s in the second members of the equations  $A(A + \overline{B}) = A$  and  $A(A + B) = A$  are equivalent to each other.

When the simple expression  $A$  arises as a result of multiplying some simple Boolean expression  $X$  by  $A$  and another operation requiring the use of this result is there, then the  $A$  should be placed at the side of  $X$  as a multiplier. If for the impending operation,  $A$ , which is either logic 0 or 1, is a divisor, then due to identical multiplier and divisor, the  $A$  (in the numerator and denominator) should be cancelled out and  $X$  results.

In a bivariable system of Boolean algebra, there are four possible simple Boolean expressions for this  $X$  which when individually multiplied by  $A$  gives the same logic form  $A$ . These are 1,  $A$ ,  $A + B$  and  $A + \overline{B}$ , for

$$\begin{aligned} 1 \times A &= A \\ A \times A &= A \\ (A + B) \times A &= A \\ (A + \overline{B}) \times A &= A. \end{aligned} \tag{3.2}$$

In the set of four equations, i.e. in (3.2), each resulting  $A$  is identical with the Boolean expression from which it is derived as indicated by the equality sign. But all the Boolean expressions in the first members of (3.2) are all



equivalent to one another because they are all equal to  $A$ . It follows also that all the resulting  $A$ s are equivalent to each other.

If we divide both sides of each equation by the multiplier  $A$ , we obtain the following.

$$\begin{aligned} 1 &= \frac{A}{A} \\ A &= \frac{A}{A} \\ A + B &= \frac{A}{A} \\ A + \bar{B} &= \frac{A}{A}. \end{aligned} \tag{3.3}$$

It can be deduced from (3.3) that the indeterminate form

$$\frac{A}{A}$$

can be Bhaskarized into four possible simple Boolean expressions, namely,  $1, A, A + B$  and  $A + \bar{B}$ .

Let us take the complements of both members of the set (3.2). This step gives

$$\begin{aligned} \overline{1 \times A} &= \bar{A} \\ \overline{A \times A} &= \bar{A} \\ \overline{(A + B) \times A} &= \bar{A} \\ \overline{(A + \bar{B}) \times A} &= \bar{A}. \end{aligned}$$

which simplifies to

$$\begin{aligned} \bar{1} + \bar{A} &= \bar{A} \\ \bar{A} + \bar{A} &= \bar{A} \\ \overline{(A + B)} + \bar{A} &= \bar{A} \\ \overline{(A + \bar{B})} + \bar{A} &= \bar{A} \end{aligned} \tag{3.4}$$

Subtracting the addend  $\bar{A}$  from both members of the set (3.4), we obtain

$$\begin{aligned} 0 &= \bar{A} - \bar{A} \\ \bar{A} &= \bar{A} - \bar{A} \\ \overline{A + B} &= \bar{A} - \bar{A} \\ \overline{A + \bar{B}} &= \bar{A} - \bar{A} \end{aligned}$$

which simplifies further to

$$\begin{aligned} 0 &= \bar{A} - \bar{A} \\ \bar{A} &= \bar{A} - \bar{A} \\ \bar{A}\bar{B} &= \bar{A} - \bar{A} \\ \bar{A}B &= \bar{A} - \bar{A}. \end{aligned}$$

From this last set of equations it is inferred that the indeterminate logic form

$$\bar{A} - \bar{A}$$

can be Bhaskarized into four Boolean expressions, namely,  $0, \bar{A}, \bar{A}\bar{B}$  and  $\bar{A}B$ .

We give another instance of Bhaskarization. We wish to find possible simple Boolean expressions for  $X$  in the equation

$$XA(A + \bar{B}) = A\bar{B}.$$

This Boolean equation is simplified to

$$XA = A\bar{B}.$$

We find  $X$  by first logically dividing both members of the above equation by the multiplier  $A$  in the left member of the equation. This gives us

$$\frac{XA}{A} = \frac{A\bar{B}}{A}.$$

The expression

$$\frac{A}{A}$$

in the first member is the logical division of  $A$  by itself and so it is equal to unity, viz

$$\frac{A}{A} = 1$$

where  $A = \{0, 1\}$ . Hence we have

$$X = \frac{A\bar{B}}{A}.$$

The second member of this last equation is indeterminate. We return to known simple Boolean expressions whose individual product with  $A$  furnishes the numerator  $A\bar{B}$  of the second member. These simple Boolean expressions are four in number and are

$$\bar{B}, \bar{A}\bar{B}, \bar{A} + \bar{B}, \text{ and } \bar{A}\bar{B} + \bar{A}B$$

for

$$A(\bar{B}) = A\bar{B}$$

$$A(\bar{A}\bar{B}) = A\bar{B}$$

$$A(\bar{A} + \bar{B}) = A\bar{B},$$

$$A(\bar{A}\bar{B} + \bar{A}B) = A\bar{B}.$$

All other Boolean expressions whose product with  $A$  give  $A\bar{B}$  are reducible to only these four simple expressions. Thus,

$$X = \frac{A(\bar{B})}{A} = \bar{B}$$

$$X = \frac{A(\bar{A}\bar{B})}{A} = \bar{A}\bar{B}$$

$$X = \frac{A(\bar{A} + \bar{B})}{A} = \bar{A} + \bar{B}$$

$$X = \frac{A(\bar{A}\bar{B} + \bar{A}B)}{A} = \bar{A}\bar{B} + \bar{A}B$$

In the above set of equations, the multiplier  $A$  and the divisor  $A$  are identical and so divide out to produce logic 1. Hence we write

$$X = \{\bar{B}, A\bar{B}, \bar{A} + \bar{B}, A\bar{B} + \bar{A}B\}$$

The instances of Bhaskarization adduced in this section are so simple that we can find the sets of Boolean quotients or differences by inspection, without using any methods. However, it is clear in more complicated cases, tabular and canonical methods may be of considerable practical importance. In the next section, we shall discuss these methods of Bhaskarizing Boolean expressions.

## 4 METHODS OF BHASKARIZING BOOLEAN EXPRESSIONS

It is within the province of this section to embrace all questions relating to what we shall call Bhaskarization methods. But before we enter into the heart of this subject, let us first consider two important aspects relating to it, namely, the complements of  $-1$  and  $1/0$  and the simplification of expressions containing  $\{0, 1\}$ .

### 4.1 Complementation of Illogical Values

When the Boolean value 1 is said to be logically subtracted from 0, or logically divided by 0, this only means the logical difference  $-1$  or the logical quotient  $1/0$  is not a logical value, but it must not by any means be thought it is impossible to form an idea of illogical values. This leads us to consider the complements of  $-1$  and  $1/0$ .

So long as we deal with only the complements of logical values, we are far from understanding the entire Boolean algebra. We must consider the complements of the illogical values, and the following lemma and theorems prepare us for discussion of these complements.

#### 4.1.1 Boolean Difference and Quotient Theorems

**Lemma 4.1.** Let  $B$  be a Boolean variable.  
 $\bar{B} - B = 1$

*Proof.* We start with De Morgan's Theorem:

$$\overline{PQ} = \overline{P + Q}.$$

If  $P$  and  $Q$  is replaced by  $B$  and  $-B$  respectively, we obtain the identity

$$\overline{B - B} = \overline{B + B}.$$

The two  $B$  s are identical to each other. Thus  $B - B = 0$  and the last identity becomes

$$\overline{0} = \overline{1}$$

which in its turn becomes the required identity,

$$\overline{0} = 1.$$

□

**Theorem 4.2.** Let  $A$  and  $B$  be Boolean variables.  
 Then

$$\overline{A - B} = \frac{\bar{A}}{B}.$$

*Proof.* We start again with De Morgan's Theorem:

$$\overline{P + Q} = \overline{PQ}.$$

Replacing  $P$  and  $Q$  with  $A$  and  $-B$  respectively, we get

$$\overline{A - B} = \overline{A + \bar{B}}.$$

By Lemma 4.1 we obtain

$$\overline{A - B} = \bar{A} \times \frac{1}{B}$$

which becomes the required identity

$$\overline{A - B} = \frac{\bar{A}}{B}.$$

□

**Theorem 4.3.** Let  $A$  and  $B$  be Boolean variables.  
 Then

$$\overline{\left(\frac{A}{B}\right)} = \bar{A} - \bar{B}.$$

*Proof.* Let us begin with Theorem 4.2:

$$\overline{A - B} = \frac{\bar{A}}{B}.$$

Replacing  $\bar{A}$  and  $\bar{B}$  with  $A$  and  $B$ , we have

$$\overline{\bar{A} - \bar{B}} = \frac{A}{B}$$

which, taking the complementation of both members of the equation, becomes the required identity

$$\overline{\left(\frac{A}{B}\right)} = \bar{A} - \bar{B}.$$

□

### 4.1.2 Complements of $-1$ and Reciprocal of Logic 0

We have here an answer to the question that at once arise in the minds of the readers. I have no doubt that by what has been written in the previous sections, the following question has arisen: What are the complements of  $-1$  and  $\frac{1}{0}$ ? The answer is given in the following theorem.

**Theorem 4.4.** *Let  $-1$  and  $\frac{1}{0}$  be resulting Boolean difference and quotient respectively in a Boolean evaluation. Then*

$$\overline{-1} = \frac{1}{0}$$

and

$$\overline{\left(\frac{1}{0}\right)} = -1$$

*Proof.* We start with Lemma 4.1:

$$\overline{-B} = \frac{1}{B}.$$

Letting  $B = 1$  gives

$$\overline{-1} = \frac{1}{1}.$$

which becomes

$$\overline{-1} = \frac{1}{0}.$$

This proves the first part of the theorem.

Take the complement of both members of this last equation. We obtain

$$-1 = \overline{\left(\frac{1}{0}\right)}$$

and the proof is finished.  $\square$

## 4.2 Simplification of Expressions Containing $\{0, 1\}$

Let  $M_1, M_2, M_3, \dots$  be minterms. We wish to simplify Boolean expressions in canonical forms with  $\{0, 1\}$  as coefficients of the minterms.

We start with the simplest case.

$$\{0, 1\}M_1 = \{0, M_1\}. \tag{4.1}$$

We now turn to the case of simplifying expressions containing two minterms with their associated  $\{0, 1\}$ .

$$\begin{aligned} \{0, 1\}M_1 + \{0, 1\}M_2 &= \{0, M_1\} + \{0, M_2\} \\ &= \{0 + 0, 0 + M_2, M_1 + 0, M_1 + M_2\} \\ &= \{0, M_2, M_1, M_1 + M_2\} \\ &= \{0, M_1, M_2, M_1 + M_2\}. \end{aligned} \tag{4.2}$$

The simplification of expressions containing three minterms with their associated  $\{0, 1\}$  is as follows.

$$\begin{aligned} \{0, 1\}M_1 + \{0, 1\}M_2 + \{0, 1\}M_3 &= \{0, M_1\} + \{0, M_2\} + \{0, M_3\} \\ &= \{0, M_1, M_2, M_1 + M_2\} + \{0, M_3\} \\ &= \{0, M_1, M_2, M_3, M_1 + M_2, M_1 \\ &\quad + M_3, M_2 + M_3, M_1 + M_2 + M_3\} \end{aligned} \tag{4.3}$$

A pattern arises. In (4.1) the resulting set has two members, 0 and the single minterm itself,  $M_1$ . In (4.2) the resulting set has four members, namely, 0, each minterm, and sum of minterms. In (4.3) the resulting set has eight members, 0, each minterm, sums of two minterms, sum of the three minterms. It follows from the pattern emerging that the  $n$ -minterm expression

$$\{0, 1\}M_1 + \{0, 1\}M_2 + \{0, 1\}M_3 + \dots + \{0, 1\}M_n$$

will result in a set consisting of  $2^n$  members which include 0, each minterm, sums of two minterms, sums of three minterms, sums of four minterms, and so on till the last member which is the sum of all the minterms. The numbers of 0, single minterms, sums of two minterms, sums of three minterms, ... sums of  $r$  minterms are respectively

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \dots, \binom{n}{r}$$

where  $\binom{n}{k}$  is a binomial coefficient.

### 4.3 Bhaskarization Techniques

Boolean expressions which are of great importance and admit of easy logical interpretation are those in which the connectives are + and  $\times$  only. Some expressions involving the connectives – and  $\div$  may be Bhaskarized into two or more expressions with only the connectives + and  $\times$ . We term these *Bhaskarizable expressions*. For instance, the expressions

$$A - AB$$

and

$$\frac{A}{A+B}$$

can be Bhaskarized into the sets of Boolean expressions  $\{A, AB\}$  and  $\{A, A+B\}$  respectively.

In the rest of this work, we shall apply two techniques in Bhaskarizing Boolean expressions, namely

1. tabular method, and
2. canonical form method.

The following example sufficiently demonstrates the connection between the two methods.

**Example 4.5.** We wish to Bhaskarize the bivariable expression  $A + B - B$  into logical expressions.

To do this, we construct a truth table for  $A + B - B$ . This is shown in Table 5.

**Table 5. Truth table for  $A + B - B$**

Row	A	B	A + B	A + B - B
1	0	0	0	0
2	0	1	1	1 - 1 = {0, 1}
3	1	0	1	1
4	1	1	1	1 - 1 = {0, 1}

From this table we obtain

$$A + B - B = (1 - 1)\bar{A}B + \bar{A}\bar{B} + (1 - 1)AB. \tag{4.4}$$

To reach this same result without resort to truth table, we only have to convert the original expression to canonical form. Thus, we get

$$\begin{aligned} A + B - B &= A(\bar{B} + B) + (\bar{A} + A)B - (\bar{A} + A)B \\ &= \bar{A}\bar{B} + \bar{A}B + \bar{A}B + AB - \bar{A}B - AB. \end{aligned}$$

We collect like terms together as follows

$$A + B - B = \bar{A}\bar{B} - \bar{A}B + \bar{A}\bar{B} + AB + AB - AB$$

which, summing  $\bar{A}B + AB$  to  $AB$ , reduces to

$$A + B - B = \bar{A}\bar{B} - \bar{A}B + \bar{A}\bar{B} + AB - AB$$

which, factorizing  $\overline{AB} - \overline{AB}$  to  $(1 - 1)\overline{AB}$  and  $AB - AB$  to  $(1 - 1)AB$ , becomes

$$A + B - B = (1 - 1)\overline{AB} + \overline{AB} + (1 - 1)AB.$$

This is the same as the result (4.4) obtained by the use of truth table.

We proceed to the completion of our Bhaskarization of the given expression. First we set  $1 - 1 = \{0, 1\}$  and work on as follows:

$$\begin{aligned} A + B - B &= (1 - 1)\overline{AB} + \overline{AB} + (1 - 1)AB \\ &= \{0, 1\}\overline{AB} + \overline{AB} + \{0, 1\}AB \\ &= \{0, \overline{AB}\} + \overline{AB} + \{0, AB\} \\ &= \{0 + \overline{AB}, \overline{AB} + \overline{AB}\} + \{0, AB\} \\ &= \{\overline{AB}, \overline{AB} + \overline{AB}\} + \{0, AB\} \\ &= \{\overline{AB} + 0, \overline{AB} + \overline{AB}, \overline{AB} + AB, \overline{AB} + 0, \overline{AB} + \overline{AB} + AB\} \\ &= \{\overline{AB}, A, \overline{AB} + \overline{AB}, A + B\} \\ &= \{A, \overline{AB}, A + B, \overline{AB} + \overline{AB}\}. \end{aligned}$$

The reader may check the resulting set by adding  $B$  to each member of the set and simplifying to see whether each simplification will yield  $A + B$ .

### 4.3.1 More Illustrative Examples

For the reader to master the art of Bhaskarization, we give more instances as follows.

**Example 4.6.** Bhaskarize the bivariable Boolean expression  $A - A\overline{B}$  into useful expressions.

To do this, we construct the logical table 6. for it.

**Table 6. Truth table for  $A - A\overline{B}$**

Row	A	B	$\overline{B}$	$A\overline{B}$	$A - A\overline{B}$
1	0	0	1	0	0
2	0	1	0	0	0
3	1	0	1	1	{0, 1}
4	1	1	0	0	1

The Bhaskarization of the expression  $A - A\overline{B}$  furnishes

$$\begin{aligned} A - A\overline{B} &= \{0, 1\}A\overline{B} + AB \\ &= \{0, A\overline{B}\} + AB \\ &= \{A\overline{B}, A\overline{B} + AB\} \\ &= \{A\overline{B}, A\} \\ &= \{A, AB\} \end{aligned}$$

**Example 4.7.** Bhaskarize  $A - AB$  into Boolean polynomial with only positive terms.

To do this, use the truth table 7.

The Bhaskarization of  $A - AB$  is

$$\begin{aligned} A - AB &= A\overline{B} + \{0, 1\}AB \\ &= A\overline{B} + \{0, AB\} \\ &= \{A\overline{B}, A\} \\ &= \{A, A\overline{B}\} \end{aligned}$$

**Table 7. Truth table for  $A - AB$**

A	B	AB	$A - AB$
0	0	0	0
0	1	0	0
1	0	0	1
1	1	1	{0, 1}

**Table 8. Truth table for  $\frac{AB}{A+B}$ .**

A	B	$\bar{A}$	$\bar{B}$	AB	$A + B$	$\frac{AB}{A+B}$
0	0	1	1	0	1	0
0	1	1	0	0	1	0
1	0	0	1	0	0	{0, 1}
1	1	0	0	1	1	1

**Example 4.8.** Bhaskarize the Boolean fraction

$$\frac{AB}{A+B}$$

We commence with the truth table 4.3.1.

From the table 4.3.1, we get the Bhaskarization as

$$\begin{aligned} \frac{AB}{A+B} &= \{0, 1\}A\bar{B} + AB \\ &= \{0, A\bar{B}\} + AB \\ &= \{AB, A\} \\ &= \{A, AB\}. \end{aligned}$$

**Example 4.9.** Bhaskarize  $\bar{P} - \bar{P}Q$ .

$$\begin{aligned} \bar{P} - \bar{P}Q &= \bar{P}(\bar{Q} + Q) - \bar{P}Q \\ &= \bar{P}\bar{Q} + \bar{P}Q - \bar{P}Q \\ &= \bar{P}\bar{Q} + \{0, 1\}\bar{P}Q \\ &= \bar{P}\bar{Q} + \{0, \bar{P}Q\} \\ &= \{\bar{P}\bar{Q} + 0, \bar{P}\bar{Q} + \bar{P}Q\} \\ &= \{\bar{P}\bar{Q}, \bar{P}\} \\ &= \{\bar{P}, \bar{P}\bar{Q}\} \end{aligned}$$

**Example 4.10.** Bhaskarize  $A + B - \bar{A}\bar{B} - \bar{A}B$ .

$$\begin{aligned} A + B - \bar{A}\bar{B} - \bar{A}B &= A(\bar{B} + B) + (\bar{A} + A)B - \bar{A}\bar{B} - \bar{A}B \\ &= \bar{A}\bar{B} + AB + \bar{A}B + AB - \bar{A}\bar{B} - \bar{A}B \\ &= AB + \bar{A}\bar{B} - \bar{A}\bar{B} + \bar{A}B - \bar{A}B \\ &= AB + \{0, 1\}\bar{A}\bar{B} + \{0, 1\}\bar{A}B \\ &= AB + \{0, \bar{A}\bar{B}\} + \{0, \bar{A}B\} \\ &= \{AB, A\} + \{0, \bar{A}B\} \\ &= \{A, B, AB, A + B\}. \end{aligned}$$

**Example 4.11.** Bhaskarize  $A(B + C) - ABC$ .

$$\begin{aligned}
 A(B + C) - ABC\bar{C} &= AB + AC - ABC\bar{C} \\
 &= AB(\bar{C} + C) + A(\bar{B} + B)C - ABC\bar{C} \\
 &= ABC + A\bar{B}C + ABC\bar{C} - ABC\bar{C} \\
 &= ABC + A\bar{B}C + \{0, 1\}ABC\bar{C} \\
 &= ABC + A\bar{B}C + \{0, ABC\bar{C}\} \\
 &= \{AC, A(B + C)\}.
 \end{aligned}$$

**Example 4.12.** Determine the possible simple set expressions for  $X$  which satisfy the set equation

$$(A' \cup B) \cap X = A \cap B.$$

To solve this set equation, we first transform it into Boolean equation, viz

$$(\bar{A} + B)X = AB.$$

Dividing both members of this equation by the multiplicand  $\bar{A} + B$  in the first member, we get

$$\frac{(\bar{A} + B)X}{\bar{A} + B} = \frac{AB}{\bar{A} + B}$$

which, understanding that

$$\frac{\bar{A} + B}{\bar{A} + B} = 1,$$

becomes

$$X = \frac{AB}{\bar{A} + B}.$$

We now apply the canonical method to Bhaskarize the second member of the equation just mentioned. We start by expressing both the numerator and denominator in canonical form. Thus we have

$$X = \frac{0\bar{A}\bar{B} + 0\bar{A}B + 0A\bar{B} + AB}{\bar{A}\bar{B} + \bar{A}B + 0A\bar{B} + AB}$$

which becomes

$$X = \frac{0}{1}\bar{A}\bar{B} + \frac{0}{1}\bar{A}B + \frac{0}{0}A\bar{B} + \frac{1}{1}AB$$

which is simplified further as follows:

$$\begin{aligned}
 X &= 0\bar{A}\bar{B} + 0\bar{A}B + \{0, 1\}A\bar{B} + AB \\
 &= \{0, 1\}A\bar{B} + AB \\
 &= \{0, A\bar{B}\} + AB \\
 &= \{AB, A\} \\
 &= \{A, AB\}.
 \end{aligned}$$

This can also be reached by first applying Theorem 4.3. This is done as follows.

$$X = \frac{AB}{\bar{A} + B} = \overline{\overline{AB} - \bar{A} + B}$$

which becomes

$$\begin{aligned}
 X &= \overline{\bar{A} + \bar{B} - A\bar{B}} \\
 &= \overline{\bar{A}\bar{B} + \bar{A}B + A\bar{B} - A\bar{B}} \\
 &= \overline{\bar{A} + \{0, A\bar{B}\}} \\
 &= \overline{\bar{A}, \bar{A} + \bar{B}} \\
 &= \{A, AB\}.
 \end{aligned}$$

Finally, we express the Boolean equation in set notation as

$$X = \{A, A \cap B\}.$$

**Example 4.13.** Bhaskarize

$$\frac{A - B}{A - \overline{AB}}$$

Let

$$f(A, B) = \frac{A - B}{A - \overline{AB}}$$

We find the coefficients of all the minterms as follows. The coefficient of  $\overline{A}\overline{B}$  is

$$f(0, 0) = \frac{0 - 0}{0 - \overline{0} \cdot \overline{0}} = \frac{0}{0} = \{0, 1\},$$

that of  $\overline{A}B$  is

$$f(0, 1) = \frac{0 - 1}{0 - \overline{0} \cdot 1} = \frac{-1}{-1} = 1,$$

that of  $A\overline{B}$  is

$$f(1, 0) = \frac{1 - 0}{1 - \overline{1} \cdot 0} = \frac{1}{1} = 1,$$

and that of  $AB$  is

$$f(1, 1) = \frac{1 - 1}{1 - \overline{1} \cdot 1} = \frac{1 - 1}{1} = \{0, 1\}.$$

The Bhaskarization of  $f(A, B)$  gives

$$\begin{aligned} f(A, B) &= f(0, 0)\overline{A}\overline{B} + f(0, 1)\overline{A}B + f(1, 0)A\overline{B} + f(1, 1)AB \\ &= \{0, 1\}\overline{A}\overline{B} + \overline{A}B + AB + \{0, 1\}AB \\ &= \{0, \overline{A}\overline{B}\} + \overline{A}B + AB + \{0, AB\} \\ &= \{0, \overline{A}\overline{B}\} + \{0, AB\} + \overline{A}B + AB \\ &= \{0, \overline{A}\overline{B}, AB, \overline{A}\overline{B} + AB\} + \overline{A}B + AB \\ &= \{1, A + B, \overline{A} + \overline{B}, \overline{A}B + AB\}. \end{aligned}$$

**Example 4.14.** Bhaskarize

$$\frac{\overline{A}}{A - B}$$

Let

$$f(A, B) = \frac{\overline{A}}{A - B}$$

We find the coefficients of all the minterms as follows. Setting  $A = 0, B = 0$ , we get

$$f(0, 0) = \frac{\overline{0}}{0 - 0} = \frac{1}{1} = 1,$$

Setting  $A = 0, B = 1$ , we get

$$f(0, 1) = \frac{\overline{0}}{0 - 1} = \frac{1}{-1}.$$

By Theorem 4.4,  $\overline{-1} = \frac{1}{0}$ , we have

$$f(0, 1) = \frac{1}{\frac{1}{0}} = 0.$$

Setting  $A = 1, B = 0$ , we get

$$f(1, 0) = \frac{\overline{1}}{1 - 0} = \frac{0}{0} = \{0, 1\}.$$

Setting  $A = 1, B = 1$ , we get

$$f(1, 1) = \frac{\overline{1}}{1 - 1} = \frac{0}{1 - 1}.$$



By Theorem 4.2 we get  $\overline{1-1} = \frac{0}{0}$ . Therefore, we have

$$f(1, 1) = \frac{0}{\frac{0}{0}} = \frac{0}{0} = \{0, 1\}.$$

The Bhaskarization of  $f(A, B)$  is

$$\begin{aligned} f(A, B) &= f(0, 0)\overline{A}\overline{B} + f(0, 1)\overline{A}B + f(1, 0)A\overline{B} + f(1, 1)AB \\ &= \overline{A}\overline{B} + 0 \cdot \overline{A}B + \{0, 1\}A\overline{B} + \{0, 1\}AB \\ &= \overline{A}\overline{B} + \{0, A\overline{B}\} + \{0, AB\} \\ &= \{\overline{B}, \overline{A}\overline{B}, A + \overline{B}, \overline{A}\overline{B} + AB\} \end{aligned}$$

## 5 APPLICATIONS IN DIGITAL CIRCUITS

We are concerned in this section with illustrative examples of how logical subtraction and division may be applied in the design of digital circuits [21], [22], [23].

**Example 5.1.** Determine possible logic forms of  $X$  in Fig. 1. if the output  $Y = A + B$ .

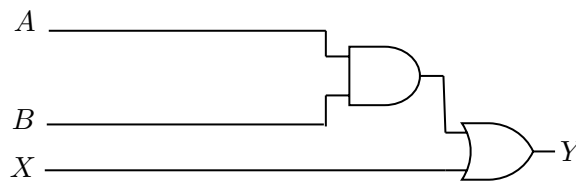


Fig. 1.

The Boolean equation for the given digital system is

$$AB + X = Y$$

which becomes

$$AB + X = A + B.$$

Expressing  $X$  in terms of  $A$  and  $B$ , we have

$$X = A + B - AB.$$

The Boolean expression on the right side of the equality sign undergoes Bhaskarization. Thus we have

$$\begin{aligned} X &= A + B - AB \\ &= A(\overline{B} + B) + (\overline{A} + A)B - AB \\ &= A\overline{B} + AB + \overline{A}B + AB - AB \\ &= A\overline{B} + \overline{A}B + AB - AB \\ &= A\overline{B} + \overline{A}B + \{0, 1\}AB \\ &= A\overline{B} + \overline{A}B + \{0, AB\} \\ &= \{A\overline{B} + \overline{A}B, A + B\} \\ &= \{A + B, A\overline{B} + \overline{A}B\}. \end{aligned}$$

**Example 5.2.** The letter  $X$  in Fig. 2. below represents the Boolean expression for a missing system of gates. Determine the logic form of  $X$  and hence replace the missing system of gates.

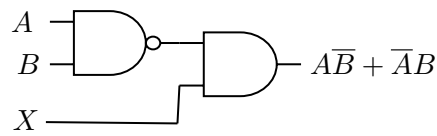


Fig. 2.

The switching equation of the AND gate is

$$X \cdot \overline{AB} = A\bar{B} + \bar{A}B$$

which becomes

$$X = \frac{A\bar{B} + \bar{A}B}{\overline{AB}}$$

This, by the application of Theorem 4.3, changes to

$$\begin{aligned} X &= \overline{\overline{A\bar{B} + \bar{A}B} - \overline{AB}} \\ &= \overline{\overline{A\bar{B}} + \overline{AB} - \overline{AB}} \\ &= \overline{\overline{A\bar{B}} + \{0, 1\}AB} \\ &= \overline{\overline{A\bar{B}} + \{0, AB\}} \\ &= \overline{\{\overline{A\bar{B}}, AB + \overline{A\bar{B}}\}} \\ &= \overline{\{\overline{A\bar{B}} + AB + \overline{A\bar{B}}\}} \\ &= \{A + B, \overline{A\bar{B}} + \overline{AB}\} \end{aligned}$$

The possible Boolean expressions for  $X$  are the logic forms  $A + B$  and  $\overline{A\bar{B}} + \overline{AB}$ . The missing digital circuit is either an OR gate or a XOR gate. Since the OR gate is simpler, we connect it to the given circuit and obtain the circuit as shown in Fig. 3. below.

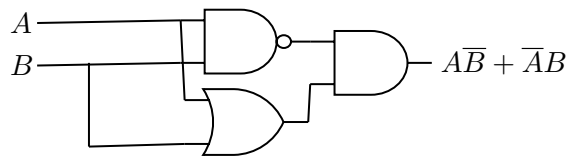


Fig. 3.

**Example 5.3.** The letter  $X$  in Fig. 4 represents the simplest possible Boolean expression for a missing switching system. Determine the Boolean expression and hence draw the complete circuit.

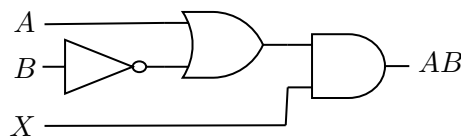


Fig. 4.

The switching equation of the switching network shown in Fig. 4. is

$$X(A + \bar{B}) = AB$$

which becomes

$$X = \frac{AB}{A + \overline{B}}$$

This, by the application of Theorem 4.3, is transformed into

$$\begin{aligned} X &= \overline{\overline{AB} - \overline{A + \overline{B}}} \\ &= \overline{\overline{A} + \overline{B} - \overline{AB}} \\ &= \overline{\overline{A}(\overline{B} + B) + (\overline{A} + A)\overline{B} - \overline{AB}} \\ &= \overline{\overline{A}\overline{B} + \overline{A}B + \overline{A}B - \overline{AB}} \\ &= \overline{\overline{B} + \{0, 1\}\overline{A}B} \\ &= \overline{\overline{B} + \{0, \overline{A}B\}} \\ &= \overline{\{\overline{B}, \overline{A} + \overline{B}\}} \\ &= \{B, AB\} \end{aligned}$$

The possible Boolean expressions for  $X$  are the logic forms  $B$  and  $AB$ . Since  $B$  (input) is simpler than  $AB$  (logic gate), we have the required complete circuit as shown in Fig. 5.

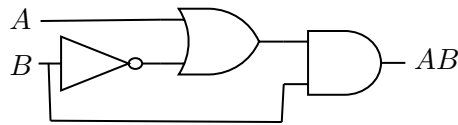


Fig. 5.

**Example 5.4.** A logic circuit which implements the function

$$F = \overline{A}BC + \overline{A}\overline{B}\overline{C} + \overline{A}BC$$

is shown in Fig. 6.  $X$  in this figure represents the simplest possible digital system. Determine  $X$  and draw the complete circuit.

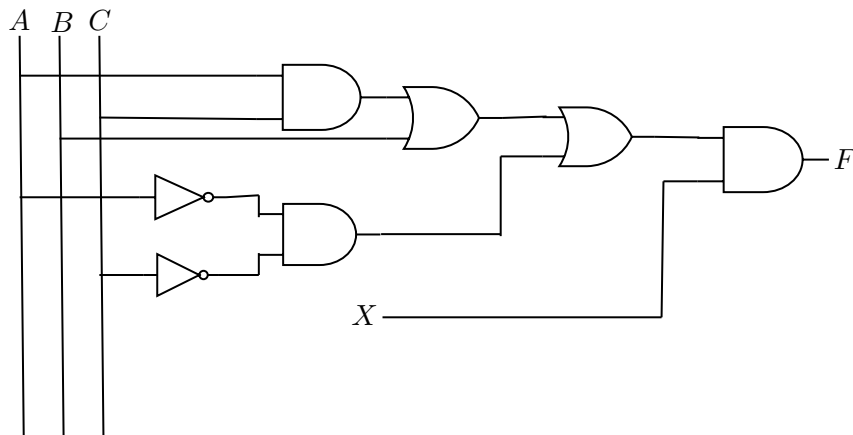


Fig. 6.

From the circuit, we get the logic equation

$$X(AC + B + \overline{A}\overline{C}) = \overline{A}\overline{B}\overline{C} + \overline{A}\overline{B}\overline{C} + \overline{A}BC.$$

This is rewritten as

$$X = \frac{\overline{A}\overline{B}\overline{C} + \overline{A}\overline{B}\overline{C} + \overline{A}BC}{AC + B + \overline{A}\overline{C}}.$$

We construct the truth table 9. which defines  $X$ . From this table we get the possible logic expressions

**Table 9. Truth table for  $X$**

A	B	C	$\overline{A}\overline{B}\overline{C} + \overline{A}\overline{B}\overline{C} + \overline{A}BC$	$AC + B + \overline{A}\overline{C}$	X
0	0	0	1	1	1
0	0	1	0	0	{0, 1}
0	1	0	0	1	0
0	1	1	1	1	1
1	0	0	0	0	{0, 1}
1	0	1	1	1	1
1	1	0	0	1	0
1	1	1	0	1	0

as follows.

$$\begin{aligned} X &= \overline{A}\overline{B}\overline{C} + \{0, 1\}\overline{A}\overline{B}\overline{C} + \overline{A}BC + \{0, 1\}\overline{A}\overline{B}\overline{C} + \overline{A}BC \\ &= \overline{A}\overline{B}\overline{C} + \{0, 1\}\overline{A}\overline{B}\overline{C} + \{0, 1\}\overline{A}\overline{B}\overline{C} + \overline{A}BC + \overline{A}BC \\ &= \overline{A}\overline{B}\overline{C} + \{0, \overline{A}\overline{B}\overline{C}, \overline{A}\overline{B}\overline{C}, \overline{A}\overline{B}\overline{C} + \overline{A}\overline{B}\overline{C}\} + \overline{A}BC + \overline{A}BC \\ &= \overline{A}\overline{B}\overline{C} + \{0, \overline{A}\overline{B}\overline{C}, \overline{A}\overline{B}\overline{C}, \overline{A}\overline{B}\overline{C} + \overline{A}\overline{B}\overline{C}\} + \overline{A}BC + \overline{A}BC \\ &= \{0 + \overline{A}\overline{B}\overline{C}, \overline{A}\overline{B}\overline{C} + \overline{A}\overline{B}\overline{C}, \overline{A}\overline{B}\overline{C} + \overline{A}\overline{B}\overline{C}, \overline{A}\overline{B}\overline{C} + \overline{A}\overline{B}\overline{C} + \overline{A}\overline{B}\overline{C}\} \\ &\quad + \overline{A}BC + \overline{A}BC \\ &= \{\overline{A}\overline{B}\overline{C} + \overline{A}BC + \overline{A}BC, \overline{A}\overline{B}\overline{C} + \overline{A}\overline{B}\overline{C} + \overline{A}BC + \overline{A}BC, \overline{A}\overline{B}\overline{C} + \overline{A}\overline{B}\overline{C} \\ &\quad + \overline{A}BC + \overline{A}BC, \overline{A}\overline{B}\overline{C} + \overline{A}\overline{B}\overline{C} + \overline{A}\overline{B}\overline{C} + \overline{A}BC + \overline{A}BC\} \end{aligned}$$

Simplifying each member of the set  $X$ , we obtain

$$X = \{\overline{A}\overline{B}\overline{C} + \overline{A}BC + \overline{A}BC, \overline{A}\overline{B} + \overline{A}\overline{C} + \overline{B}\overline{C}, \overline{A}\overline{B} + \overline{B}\overline{C} + \overline{A}BC, \overline{B} + \overline{A}\overline{C}\}$$

The simplest of these members of  $X$  is  $\overline{B} + \overline{A}\overline{C}$  and the digital system representing this is inserted in Fig. 6. to give rise to the entire digital circuit shown in Fig. 7.

**Example 5.5.** In the network of Fig. 8, determine the simplest Boolean pair  $(X, Y)$  expressed in terms of  $A$  AND  $B$ .

From the analysis of the network, we get the two equations

$$\overline{\overline{A} + X} = AB \tag{5.1}$$

and

$$\overline{AB + Y} = \overline{A}\overline{B} + \overline{A}\overline{B}. \tag{5.2}$$

We begin with the determination of  $X$  in eq. (5.1). Taking the complement of both sides gives us

$$\overline{\overline{A} + X} = \overline{A}\overline{B}$$

which becomes

$$\overline{\overline{A} + X} = \overline{A} + \overline{B}$$

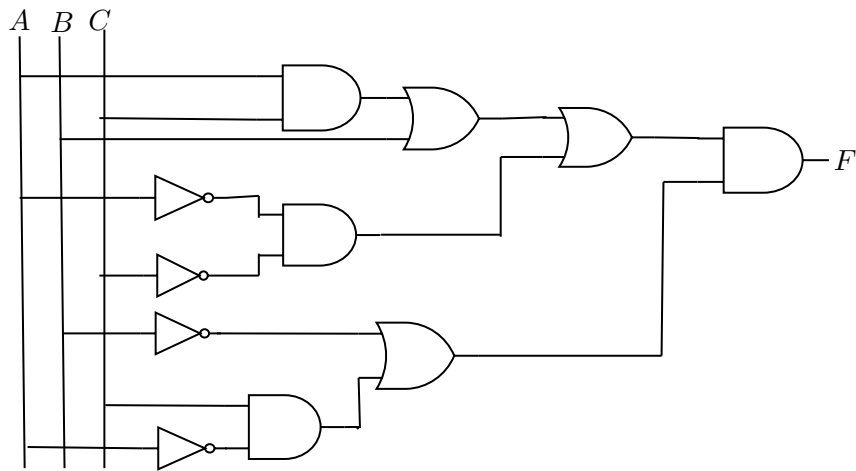


Fig. 7.

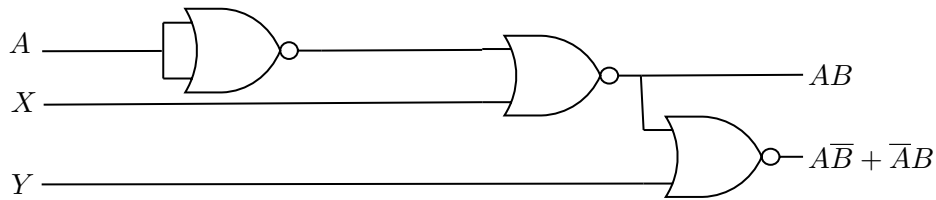


Fig. 8.

which by method of transposition gives

$$X = \overline{A} + \overline{B} - \overline{A}.$$

We Bhaskarizing the expression on the right-hand side of this last equation:

$$\begin{aligned} X &= \overline{A} + \overline{B} - \overline{A} \\ &= \overline{A}(B + \overline{B}) + (A + \overline{A})B - \overline{A}(B + \overline{B}) \\ &= (\overline{A}\overline{B} + \overline{A}B + AB) - (\overline{A}B + \overline{A}\overline{B}) \\ &= AB + \overline{A}B - \overline{A}B + \overline{A}\overline{B} - \overline{A}\overline{B} \\ &= AB + \{0, \overline{A}B\} + \{0, \overline{A}\overline{B}\} \\ &= \{\overline{B}, \overline{A}B, \overline{A} + \overline{B}, \overline{A}\overline{B} + \overline{A}B\}. \end{aligned}$$

The simplest expression for  $X$  is  $\overline{B}$ .

We proceed to the determination of the expressions for  $Y$ . From eq. (5.2) we have

$$AB + Y = \overline{\overline{A}B + \overline{A}\overline{B}}.$$

This becomes

$$\begin{aligned} Y &= \overline{\overline{A}B + \overline{A}\overline{B}} - AB \\ &= \overline{\overline{A}\overline{B}} + AB - AB \\ &= \overline{\overline{A}\overline{B}} + \{0, AB\} \\ &= \{\overline{\overline{A}\overline{B}}, \overline{\overline{A}\overline{B}} + AB\}. \end{aligned}$$

The simpler Boolean expression for  $Y$  is  $\overline{\overline{A}\overline{B}}$ . Thus the required simplest Boolean pair is  $(\overline{B}, \overline{\overline{A}\overline{B}})$ .

## 6 CONCLUSION

The notions of Boolean subtraction and division were banished from mathematical discussions because they were supposed to lack logical interpretations as well as consistencies in their use. In this paper we clarified these notions and demonstrated their usefulness in the design of digital circuits.

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## COMPETING INTERESTS

Author has declared that no competing interests exist.

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