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Every Strongly Remotal Subset In Banach Spaces is a Singleton

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Abstract

Let X be a Banach space, and $E \subset X$ be a non-empty closed bounded subset of X. The set E is called proximinal in X if for all $x \in X$ there is some $e \in E$ such that $\|x - e\| = \inf\{\|x - y\| : y \in E\}$. E is called remotal in X if for all $x \in X$, there exists $e \in E$ such that $\|x - e\| = \sup\{\|x - y\| : y \in E\}$. The concept of strong proximinality is well known by now in the literature, and many results were obtained. In this paper we introduce the concept of strong remotality of sets. Many results are presented.

Key words and Phrases: Remotal sets; strongly remotal sets.
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1 Introduction

Let X be a Banach space, and $E \subset X$ be a non-empty closed bounded subset of X.For $x \in X$, we let $D(x, E) = \sup\{\|x - e\| : e \in E\}$, and $d(x, E) = \inf\{\|x - e\| : e \in E\}$. The set E is called remotal

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in X if for all $x \in X$ there exists some $e \in E$ such that $D(x, E) = \|x - e\|$. If e is unique then E is called uniquely remotal. The set E is called proximinal if for all $x \in X$ there exists some $\theta \in E$ such that $d(x, E) = \|x - \theta\|$ .Proximinal sets have its applications in many branches of science, while remotal sets have its applications in geometry of Banach spaces. So much work has been done on proximinal sets. However remotality in Banach spaces is still in its early stages. One of the main open problems in remotality theory is "Must every uniquely remotal set in a Banach space X be a singleton". This conjecture has been open for more than 40 years. Many papers studying such conjecture has been published. It is known that this conjecture is equivalent to the conjecture "Every uniquely proximinal set in a Hilbert space is convex". This paper deals with this conjecture. We refer to [1,2], and [3] for results on this conjecture.

Through out this paper, X, is a Banach spaces. The closed unit ball of X is denoted by $B_1[X]$, and the dual of X is denoted by X^*. The Banach space of $p-$integrable functions(equivalence classes) from the compact interval $I \subset R$ into the Banach space X is denoted by $L^p(I, X)$, where R is the set of real numbers. For a non-empty set $E \subset X$, we let $L^p(I, E) = \{f \in L^p(I, X) : f(t) \in E$ $a.e.t\}$. We refer to [1,2,4], and [3] for general results on remotal sets.

1.1 Remotality and summands

Definition 1.1. Let X be a Banach space. Consider the following functions:

(i) $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(x)$ is strictly increasing, $\varphi(0) = 0$. Let M denote the class of all such functions φ.

(ii) $\psi : X \to (0, \infty)$ such that $\psi(x)$ is increasing, in the sense $\|x\| < \|y\|$ implies $\psi(x) \le \psi(y)$. Let N denote the class of all such functions ψ.

Now, let Y be a closed subspace of the Banach space X, $\varphi \in M$, and $\psi \in N$. The subspace Y is called $(\varphi, \psi)-$ **summand** in X if there is a closed subspace $W \subseteq X$ such that $X = Y \oplus W$, and whenever $x \in X$ with $x = y + w$, $y \in Y$ and $w \in W$, we have $\varphi(\| x \|) = \varphi(\| y \|) + \psi(y)\varphi(\| w \|)$.

Remark 1.2. If $\varphi(t) = t^p$, $1 \le p < \infty$, and $\psi(x) = 1$ for all $x \in X$, then $(\varphi, \psi)-$ summand subspaces in $L^p(I, X)$ are the $p-$summand subspaces.

Theorem 1.3. *Let Y be a $(\varphi, \psi)-$ summand subspace of a Banach space X. Then $B_1[Y]$ is remotal in X.*

Proof. Suppose that $X = Y \oplus W$ so that $x = y + w$. This implies that

$$\varphi(\|x\|) = \varphi(\|y\|) + \psi(y)\varphi(\|w\|)$$

Here $x \in X$, $y \in Y$ and $w \in W$. Let $\widehat{y}$ be a farthest element in $B_1[Y]$ from y. This element exists since $B_1[Y]$ is remotal in Y. Then,

$$\begin{aligned}\varphi(\|x - \widehat{y}\|) =& \varphi(\|y - \widehat{y} + w\|) \\ =& \varphi(\|y - \widehat{y}\|) + \psi(y - \widehat{y})\varphi(\|w\|)\end{aligned}$$

Since ψ is increasing and $\|y - \widehat{y}\| \ge \|y - z\|$ $\forall z \in B_1[Y]$, we get

$$\begin{aligned}\varphi(\|x - \widehat{y}\|) \ge& \varphi(\|y - z\|) + \psi(y - z)\varphi(\|w\|) \qquad \forall z \in B_1[Y], \\ =& \varphi(\|y + w - z\|), \qquad \forall z \in B_1[Y] \\ =& \varphi(\|x - z\|) \qquad \forall z \in B_1[Y].\end{aligned}$$

Then $\|x - \widehat{y}\| \ge \|x - z\|$ $\forall z \in B_1[Y]$, since φ is strictly increasing.
Consequently, $\widehat{y}$ is the farthest element in $B_1[Y]$ from $x = y + w$. ☐

As a consequence of Theorem 1.3 and Remark 1.2, we have:

Corollary 1.4. *The unit ball of every p-summand subspace of any Banach space is remotal.*

2 Strong Remotality

Strong proximinality was defined in many ways, in many papers. We refer to [5], and the references there in for such concept. In this section we introduce the concept of strong remotality in Banach spaces. We did get some results concerning strong proximinality.

The main result is " Every strongly remotal set is a singleton".

Throughout the paper, we don't need to assume the convexity of the set E, which we want to study its strong remotality in X. However using results in [6], we can assume the convexity of the set E.

Definition 2.1. Let E be a non-empty closed convex bounded set of the Banach space X. E is called **strongly remotal** in X if there exist $\varphi \in M$ and $\psi \in N$ with $\inf \psi(y) > 0$ such that for all $x \in X$ there exists some $y \in E$ such that, for all $z \in E$,

$$\varphi\|x - y\| \geq \varphi\|x - z\| + \psi(y)\,\varphi\|y - z\|.$$

An equivalent form of strong remotality is:

Proposition 2.2. Let E be a non-empty closed convex set of the Banach space X. Then the following are equivalent.

(i) E is strongly remotal set in X with the associated functions φ and ψ.

(ii) For every $x \in X$ there exists $y \in E$ for which

$$\inf\left\{\frac{\phi\|x-y\| - \phi\|x-z\|}{\phi\|y-z\|} : z \in E\backslash\{y\}\right\} > 0.$$

Proof. $(ii) \to (i)$: Let $\inf\left\{\frac{\phi\|x-y\| - \phi\|x-z\|}{\phi\|y-z\|} : z \in E\backslash\{y\}\right\} = \epsilon > 0$. Then define $\epsilon = \psi(y)$, so

$$\phi\|x-y\| - \phi\|x-z\| = \psi(y)\,\phi\|y-z\|.$$

Hence, E is strongly remotal in X.

Conversely, $(i) \to (ii)$. Let E be strongly remotal in X. Hence, there exist two functions φ and ψ such that

$$\varphi\|x - y\| \geq \varphi\|x - z\| + \psi(y)\,\varphi\|y - z\|$$

Thus, $\frac{\varphi\|x-y\| - \varphi\|x-z\|}{\varphi\|y-z\|} \geq \psi(y)$ for all $z \in G\backslash\{y\}$. But $\inf \psi(y) > 0$, so

$$\inf\left\{\frac{\varphi\|x-y\| - \varphi\|x-z\|}{\varphi\|y-z\|} : z \in E\backslash\{y\}\right\} > 0.$$

□

Theorem 2.3. *Let E be a strongly remotal subset of X with associated maps φ and ψ. Then E is uniquely remotal.*

Proof. Let $x \in X$ be arbitrary. Then from Definition 2.1 there exists $y \in E$ such that, for all $z \in E$, we have:

$$\varphi\|x-y\| \geq \varphi\|x-z\| + \psi(y)\varphi\|y-z\|.$$

Since $\psi(y) > 0$ and $\varphi(r) > 0$ for $r > 0$, we get

$$\varphi\|x-y\| > \varphi\|x-z\|, \qquad \forall\ z(\neq y) \in E.$$

But φ is strictly increasing. It follows that

$$\|x-y\| > \|x-z\|, \qquad \forall z(\neq y) \in E.$$

Hence, y is the only farthest point from x, and E is uniquely remotal. □

Now, for a non-empty closed convex set $E \subset X$, we let $P(E)$ denote the power set of E. The map $F(.,E) : X \to P(E)$, defined by $F(x,E) = \{e \in E : \|x-e\| = D(x,E)\}$ is called the metric projection associated with E. If E is uniquely remotal the map $F(x,E)$ is a function, and we write $F(x)$ for $F(x,E)$. In general, $F(x)$ need not be continuous.

The next result states that the farthest point map $F(x)$ is continuous for strongly remotal sets in normed spaces.

Theorem 2.4. *Let E be a non-empty strongly remotal subset of X with associated maps φ and ψ. Then the map $F : X \to E$, $F(x) = F(x,E)$ is continuous at each x for which $r = D(x,E)$ is a point of continuity of φ.*

Proof. Since φ is strictly increasing, φ is continuous on $[0,\infty)$ except on a set E of Lebesgue measure zero. Let $x \in X$ such that $D(x,E) = r$ is a point of continuity of φ. Let (x_n) be a sequence in X such that $x_n \to x$, and $D(x_n,E) = r_n$. We claim that $F(x_n) \to F(x)$. Now, since E is strongly remotal, then we have

$$\varphi\|x - F(x)\| \geq \varphi\|x-z\| + \psi(z)\varphi\|F(x) - z\|$$

for all $z \in E$. Take $z_n = F(x_n)$. Hence,

$$\varphi\|x - F(x)\| \geq \varphi\|x - F(x_n)\| + \psi(y)\varphi\|F(x) - F(x_n)\| \text{ for all } n.$$

This implies

$$\varphi\|F(x) - F(x_n)\| \leq \frac{1}{\psi(y)}[\,\varphi\|x - F(x)\| - \varphi\|x - F(x_n)\|],$$

But $\|x - F(x)\| = r$. Then

$$\varphi\|F(x) - F(x_n)\| \leq \frac{1}{\psi(y)}[\,\varphi(r) - \varphi\|x - F(x_n)\|] \tag{2.1}$$

Now,

$$\|x_n - F(x)\| \leq \|x_n - F(x_n)\| \leq \|x_n - x + x - F(x_n)\| \leq \|x_n - x\| + \|x - F(x_n)\|.$$

Hence, taking the limits, we get

$$\|x - F(x)\| \leq \lim \|x - F(x_n)\|. \tag{2.2}$$

Now,

$$\|x - F(x_n)\| \leq \|x - x_n + x_n - F(x_n)\| \leq \|x - x_n\| + \|x_n - F(x_n)\|.$$

Hence, by taking limits we get

$$\lim \|x - F(x_n)\| \leq \lim \|x_n - F(x_n)\|.$$

But, since $x_n \to x$, then see [4], $D(x_n, E) \to D(x, E)$. Thus, $\lim ||x_n - F(x_n)|| = ||x - F(x)||$. This implies

$$\lim \|x - F(x_n)\| \leq \|x - F(x)\| . \tag{2.3}$$

It follows from (2.2) and (2.3), $\|x - F(x)\| = \lim \|x - F(x_n)\|$.

Equation (2.1) gives

$$\lim \varphi \|F(x) - F(x_n)\| \leq \frac{1}{\psi(y)} [\varphi \|x - F(x)\| - \lim \varphi \|x - F(x_n)\|].$$

But φ is continuous at r. Hence $\lim \varphi \|F(x) - F(x_n)\| = 0$. Since φ is strictly increasing and $\varphi(0) = 0$, it follows that $\lim \|F(x) - F(x_n)\| = 0$.

This ends the proof of the theorem. □

Now, we have

Corollary 2.5. *If φ is continuous on $[0, \infty)$, then the function $F(x)$ is continuous on X.*

Let $C(I, X), L^p(I, X), L^\infty(I, X)$ be the classical vector valued continuous, Lebesgue p-integrable, and essentially bounded functions defined on the compact interval I, with values in the Banach space X. As a consequence of Theorem 2.4, we have the following:

Theorem 2.6. *Let E be a strongly remotal subset of the Banach space X, with continuous φ. Then*

1. *$C(I, E)$ is remotal in $C(I, X)$.*

2. *$L^p(I, E)$ is remotal in $L^p(I, X)$.*

3. *$L^\infty(I, E)$ is remotal in $L^\infty(I, X)$.*

Proof. (1) Let $f \in C(I, X)$ then $f(t) \in X \; \forall t \in I$. Since E is strongly remotal, then by Theorem 2.4, the function $F(x)$ is continuous on X with values in E.

Then $F \circ f \in C(I, E)$ since it is the composition of continuous functions. But, $F \circ f(t)$ is the farthest point from $f(t)$ in E. Now,

$$\begin{aligned} \|f - F \circ f\|_\infty &= \sup_t \|f(t) - F \circ f(t)\| \\ &\geq \sup_t \|f(t) - g(t)\|, \forall g(t) \in E. \\ &= \|f - g\|_\infty, \quad \forall g \in C(I, E). \end{aligned}$$

Hence $\|f - g\|_\infty \leq \|f - F \circ f\|_\infty$.

So, $C(I, E)$ is remotal in $C(I, X)$, and $F \circ f \in F(f, C(I, E))$.

(2) Let $f \in L^p(I, X)$. Then $f(t) \in X$. Therefore, $F \circ f \in L^p(I, E)$, since F is continuous and E is a bounded set such that $F \circ f(t)$ is the farthest point from $f(t)$ in E. Now,

$$\begin{aligned} \|f - F \circ f\|_p^p &= \int_I \|f(t) - F \circ f(t)\|^p dt \\ &\geq \int_I \|f(t) - g(t)\|^p, \forall g(t) \in E. \\ &= \|f - g\|_p^p, \quad \forall g \in L^p(I, E). \end{aligned}$$

Hence, $\|f-g\|_p \leq \|f-F\circ f\|_p$.
So, $L^p(I,E)$ is remotal in $L^p(I,X)$, and $F\circ f \in F(f, L^p(I,E))$.

(3) Let $f \in L^\infty(I,X)$. Then $f(t) \in X$.
Then $F\circ f \in L^\infty(I,G)$, since F is continuous and E is a bounded set such that $F\circ f(t)$is the farthest point from $f(t)$ in E. Now,

$$\begin{aligned}\|f-F\circ f\|_\infty &= ess\sup_t \|f(t)-F\circ f(t)\| \\ &\geq ess\sup_t \|f(t)-g(t)\|, \forall g(t)\in E. \\ &= \|f-g\|_\infty, \quad \forall g \in L^\infty(I,E).\end{aligned}$$

Hence $\|f-g\|_\infty \leq \|f-F\circ f\|_\infty$.
So, $L^\infty(I,E)$ is remotal in $L^\infty(I,X)$.

□

Corollary 2.7. *Let X be a Banach space and E be a closed bounded subset of X. Then, for $1\leq p<\infty$, we have $g\in L^P(I,E)$ is the farthest point from $f\in L^P(I,X)$ if and only if, for almost all $t\in I$, $g(t)$ is the farthest point in E from $f(t)$.*

Theorem 2.8. Suppose that $L^P(I,E)$ is strongly remotal in $L^P(I,X)$ for some $1\leq p<\infty$, where E is a non-empty closed bounded subset of a Banach space X. Then E itself is strongly remotal in X.

Proof. Let $x\in X$ be arbitrary, and define $f(t)=x$. Then, being a constant function, $f\in L^P(I,X)$. Therefore, there exists $g\in L^P(I,E)$ and two functions ψ and φ such that

$$\varphi\|f-g\|_p \geq \varphi\|f-h\|_p + \psi(g)\,\varphi\|g-h\|_p \text{ for all } h\in L^P(I,E).$$

By Corollary 2.7 $\|f(t)-g(t)\| \geq \|f(t)-h(t)\|, \forall e\in E$, for almost all $t\in I$. Since f is the constant function x, then g is a constant function say, $g(t)=y$. Hence

$$\varphi\|x-y\|_p \geq \varphi\|x-h\|_p + \psi(y)\,\varphi\|y-h\|_p \text{ for all } h\in L^P(I,E).$$

Choosing h to be the constant function $h(t)=z$, then, $\varphi\|x-y\| \geq \varphi\|x-z\| + \psi(y)\,\varphi\|y-z\|$ for all $z\in E$. □

As for the classical sequence spaces we have the following results

Theorem 2.9. Let E be a finite set in a Banach space X. Then $\ell^p(E)$ is remotal in $\ell^p(X)$ if and only if $E=\{0\}$.

Proof. Since E is finite, E is remotal. If $E=\{0\}$, then $\ell^p(E)=\{0\}$ is remotal in $\ell^p(X)$. If $0\neq e\in E$, then $y=(e,e,e,e,e,....,e,0,0,0,0....)\in\ell^p(E)$, where e appears in the first $n-$coordinates.

but the taking $x=0\in\ell^p(X)$, we have $D(x,\ell^p(E))\geq n\,\|e\|$. Since n is arbitrary, the result follows. □

One can easily prove:

Theorem 2.10. *Let E be a finite set in X that contains 0. Then $c_0(E)$ is remotal in $c_0(X)$.*

We end this section with the following questions

Problem 1. Every uniquely remotal set in a Banach space X is strongly remotal. An Affirmative answer to this question solves the classical conjecture mentioned at the beginning of this paper affirmatively.

Problem 2. If E is strongly remotal in X, must $L^p(I,E)$ be strongly remotal in $L^p(I,X)$?

3 Conclusion

We have proved that every strongly remotal set in a Banach space is a singleton.

Competing Interests

The authors declare that no competing interests exist.

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